The Creation of a Humanistic Educational Framework for the Nature of Pure Mathematics

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Within the field of mathematics education research, scholars have found that students often have naïve views about the nature of mathematics. Mathematics is seen as an impersonal and uncreative subject. What can educators do to challenge such views, and support students in developing richer understandings of the nature of mathematics? In this paper, I describe my dissertation study, the goal of which was to identify humanistic characteristics of pure mathematics which may be of benefit for undergraduate students in a transition-to-proof course to know and understand. Using the methodological framework of heuristic inquiry, which leverages the researcher as instrument in qualitative research, I identified humanistic characteristics of mathematics by reviewing relevant literature, collaborating with a professional mathematician, co-teaching an undergraduate transition-to-proof course, and being open to mathematics wherever it appeared in life. The main result is the IDEA Framework for the Nature of Pure Mathematics.

Keywords: Nature of Mathematics, Identity, Dynamic Knowledge, Exploration, Argumentation

Students rarely have an opportunity to reflect on the nature of mathematics. Many have naïve views of mathematics, perhaps believing that mathematics is a static body of knowledge consisting of arbitrary rules and procedures (Beswick, 2012; Erlwanger, 1973; Muis, Trevors, Duffy, Ranellucci, & Foy, 2016; Presmeg, 2007; Solomon & Croft, 2016; Thompson, 1992). These naïve views may negatively affect the learning of mathematics (Erlwanger, 1973; Maciejewski, 2016). As Maciejewski (2016) claimed, "A deeper, connected view of the subject correlates to a deeper approach to study […] Fragmented, superficial perspectives often result in less desirable outcomes" (p. 1). Many mathematics education scholars view and describe mathematical knowledge as a dynamic human product (Boaler, 2016), and emphasize the human aspects of mathematical work such as creativity (Burton, 1999) and fallibility (Ernest, 1991). These modern views are influenced by cultural approaches to mathematics (Bishop, 1988), theories of embodied cognition (Lakoff & Nuñez, 2000), humanistic philosophy of mathematics (Ernest, 1991), or perhaps scholars' own experiences doing mathematical work (e.g. Hersh, 1997). The gap between the views of mathematics held by students and the perspectives held by scholars needs to be addressed within mathematics education research.

Purpose of the Study

While scholars in science education have done significant research aimed at understanding the teaching and learning of the nature of science (Lederman & Lederman, 2014), including undergraduate research (e.g. Abd-El-Khalick & Lederman, 2000; Schalk, 2012; Willoughby & Johnson, 2017), relatively little research has been done on this subject within mathematics education (Kean, 2012; Jankvist, 2015; White-Fredette, 2010). Research on the teaching and learning of the nature of science (NOS) is guided by frameworks or lists that explicitly outline goals for students' understanding of NOS (Lederman & Lederman, 2014). For instance, a goal is for students to understand that "Scientific knowledge is open to revision in light of new evidence" (NGSS, 2013, p. 4). These lists aid researchers in assessing whether instruction is effective in teaching students about the nature of science.

Researchers in mathematics education have not systematically studied the nature of mathematics to the extent that science education researchers have studied NOS (Kean, 2012). Our field has lists that outline important mathematical practices (e.g. CCSSI, 2010; NCTM, 2000) and mathematical habits of mind (e.g. Cuoco, Goldenberg, & Mark, 1996), but we do not have lists that outline goals for students' understanding of the nature of mathematics. Such a list would provide university instructors a guide for teaching the nature of mathematics to undergraduates mathematics students, including pre-service teachers. Alba Thompson (1992) noted, "Very few cases of teachers with an informed historical and philosophical perspective of mathematics have been documented in the literature" (p. 141). School teachers will not have informed views until the university, the place where teachers are educated, makes the nature of mathematics a subject of study for its students.

The purpose of this research project was to produce a humanistic framework for the nature of mathematics outlining characteristics of mathematics that may serve as goals for undergraduates' understandings. Two broad questions, "What is the nature of pure mathematics?" and "What should students understand about the nature of pure mathematics?" guided this study. Moreover, I focused on undergraduate students' understanding of the nature of pure mathematics within a transition-to-proof course. I sought to understand, "What should undergraduate students in a transition-to-proof course understand about the nature of pure mathematics?"

Felix Browder (1976) defined pure mathematics to be "that part of mathematical activity that is done without explicit or immediate consideration of direct application to other intellectual domains or domains of human practice" (p. 542). Undergraduate mathematics majors and minors experience pure mathematics in courses such as abstract algebra, topology, analysis, and transition-to-proof. Within transition-to-proof courses, students are expected to pick up the terminology of pure mathematics (e.g. theorem, conjecture, proof), learn to write proofs, and develop an understanding of selected pure mathematics content (e.g. set theory, functions and relations). To meet these learning goals, it may be necessary for instructors to discuss pure mathematics' particular nature, because what is valued in transition-to-proof may be different than what has been valued in students' prior mathematics courses.

Methodology

Theoretical and Methodological Frameworks

I sought to understand what is the nature of pure mathematics? But of course, pure mathematics is what mathematicians do. Courant and Robbins (1941) wrote, "For scholars and laymen alike it is not philosophy but active experience in mathematics itself that can alone answer the question: What is mathematics?" (p. xix). I reasoned that if I really wanted to understand the nature of mathematics, then I must have experience doing mathematics. I thus decided that a core feature of my study would be the documentation of and reflection on my collaboration with a research mathematician. Patton (2015) wrote that the core question of heuristic inquiry is "What is my experience of this phenomenon and the essential experience of others who also experience this phenomenon intensely?" (p. 118). In this light, heuristic inquiry seemed to be a perfect fit to study my experience doing pure mathematics for the purposes of developing a humanistic educational framework for the nature of mathematics. Heuristic inquiry is a self-study, and Douglass and Moustakas (1985) noted that, "It is the focus on the human person in experience and that person's reflective search, awareness, and discovery that constitutes the essential core of heuristic investigation" (p. 42). The ultimate end of heuristic inquiry is what Moustakas (1990) called the creative synthesis, in which

> The researcher creates an original integration of the material that reflects the researcher's intuition, imagination, and personal knowledge of meanings and essences of the experience. The creative synthesis may take the form of a lyric poem, a song, a narrative description, a story, or a metaphoric tale. In this way, the experience as a whole is presented, and, unlike most research studies, the individual persons remain intact. (p. 51)

Narratives play an important role in the mathematics education research (e.g. Ball, 1993; Erlwanger, 1973; Lampert, 1990), as stories can provide context for discussing and reflecting on ideas. In addition to a humanistic framework for the nature of mathematics (presented in the results of this paper), my dissertation also features ten stories that illuminate the characteristics of mathematics that comprise the framework. These are stories of my collaboration with a professional mathematician, events that took place in a transition-to-proof classroom I co-taught, or perhaps meaningful stories of my own family's interaction with mathematics. Each of these stories features direct quotations from the data that I collected.

The methodological framework of heuristic inquiry, which has roots in humanistic psychology, meshes well with the theoretical stance of humanism which I also take in this study in regards to the nature of mathematics. Humanistic philosophers of mathematics (e.g. Lakatos, 1976; Tymoczko, 1988) are frequently cited in mathematics education literature (e.g. Ball, 1988; Boaler, 2016; Komatsu, 2016; Lampert, 1990; Larsen & Zandieh, 2008; Weber, Inglis, Mejia-Ramos, 2014). Humanistic approaches are unique in that they take as foundational the notion that mathematical knowledge is a human product. As Hersh (1997) wrote, "To the humanist, mathematics is *ours*—our tool, our plaything" (p. 60). I sought to create a humanistic educational framework for the nature of mathematics that may guide the teaching and learning of the nature of mathematics and challenge naïve views. Humanistic philosophy of mathematics (e.g. Ernest, 1991; Hersh, 1997; Lakatos, 1976) and relevant mathematics education literature (e.g. Lampert, 1990, Thompson, 1992, White-Fredette, 2010) informed an initial review of the literature in which I identified several possible goals for student understanding of the nature of mathematics (Author, 2017). After the completion of this literature review, I continued my dissertation study using the methodological framework of heuristic inquiry.

Data Sources

In efforts to understand the nature of pure mathematics, I sought collaboration with a graph theorist, a full professor and active research mathematician, whom I refer to as Dr. Combinatorial. Dr. Combinatorial and I worked together in efforts to prove one of his unsolved conjectures related to the chromatic number of a graph. I recorded all of our conversations in which we discussed the conjecture, and kept hard copies or photos of all of our mathematical work. Throughout the process of working on the conjecture, I was not only doing mathematics, but I was constantly reflecting on my own experience and the nature of pure mathematics.

In order to reflect on what undergraduates should understand about the nature of pure mathematics, I also collected data in a transition-to-proof course required of undergraduate mathematics majors at a large Southeastern university. The course is called "Foundations of Higher Mathematics" and is meant to serve as a transition course as students proceed from lower-level to upper-level mathematics coursework. The transition represents a shift from the traditional procedurally-based school mathematics to the work that more closely resembles that of pure mathematicians. I co-taught this course with another mathematics education scholar, Dr. Amicable, who had designed the course and taught it for seven prior semesters. I fully took over teaching the last month of the semester as she took a planned leave of absence. The course was inquiry-based in nature, and students were constantly working together to draft arguments, critique arguments, and discuss and debate proof writing techniques. Twenty-three students from the course agreed to participate in the study. Dr. Amicable asked all of the students to choose a number type that best captured their own personalities. I have chosen these number types (e.g. Binary, Whole, Natural) to be their pseudonyms in this paper. I chose the number type Surreal as my own pseudonym. The data I gathered from this course included audio recordings of discussions I had with the co-instructor, audio of whole-class discussions, student homework, classwork, exit tickets, and all other class materials.

Another crucial piece of data for this self-study was a personal journal that I kept in order to write and reflect about my experiences doing and teaching mathematics. My writings were particularly focused on documenting and reflecting on my experiences relevant to the nature of mathematics (NOM) and its teaching and learning. Another source of data came from audio recordings of informal coffee-shop style interviews that I conducted with persons whom I was interested in speaking to about NOM (e.g. mathematicians). These interviews generally consisted of conversations about NOM and interviewees' opinions about what students should understand about NOM. Six people agreed to such interviews, and in some cases multiple interviews were conducted. Most notably among these were two mathematicians. Speaking to these mathematicians, I was able to get feedback on my ideas about possible goals for students' understanding of the nature of mathematics. See Table 1 for a list of all the data that was collected for this study.

Table 1. Data Sources

Mathematics Collaboration Data
Audio-recordings of discussions with mathematician
Hard copies of mathematical work (whiteboard photos and personal notebooks)
Mathematics Course Data
Class materials (e.g. handouts, PowerPoint slides)
Audio recordings of whole class discussions
Audio recordings of discussions with co-instructor
Student homework, classwork, and exit tickets
Journal Data
Journal in which the researcher reflected on his experiences doing mathematics, teaching mathematics, discussing NOM, and reading NOM literature
Other Data
Informal Interviews
Personal Audio / Other Photos / Documents / Notes

Data Analysis

Moustakas (1990) wrote that heuristic analysis is on-going from the beginning to the end of an inquiry. Throughout the data collection process, I had in mind the inquiry questions, "What is the nature of pure mathematics?" and "What should students understand about the nature of pure mathematics (NOM)?" Whenever I had an idea for a possible NOM goal (for student understanding), I wrote it out and then saved it into a single word document. At the end of data collection I had a list of fifteen possible candidates for a NOM framework in addition to the initial characteristics identified in the literature review for a total of nineteen characteristics.

Often these characteristics were the topics of conversation during the informal interviews, as I asked mathematicians and others if they considered these characteristics to be worthy goals for student understanding of the nature of mathematics.

After the data was collected, I received feedback on preliminary results at research conferences and job presentations. I then transcribed all of the data (frequently making reflective notes pertaining to the nature of mathematics), and coded the entire set of data using the qualitative software Atlas-ti according to the potential NOM characteristics, which were used as deductive codes (Patton, 2015). Based on the collected data quotations associated with each code, I drafted stories of my experience to illuminate key features of the nature of mathematics. I sought to identify features of the nature of mathematics for which I could tell clear and compelling stories; characteristics that were not only grounded in the data, but also representative of my experience.

Results

The IDEA Framework for the Nature of Mathematics

The main result of this study is the IDEA Framework for the Nature of Pure Mathematics which consists of four characteristics: 1) Our mathematical ideas and practices are part of our ***i**dentity*; 2) Mathematical knowledge and practices are ***d**ynamic* and forever refined; 3) Pure mathematical inquiry is an emotional ***e**xploration* of ideas; and 4) Mathematical ideas and knowledge are socially vetted through ***a**rgumentation*. Note that IDEA corresponds to the key concepts of each of the four characteristics: I-Identity, D-Dynamic, E-Exploration, and A-Argumentation. I also tell ten stories to illuminate these characteristics of the nature of mathematics, but due to space limitations I will only present two abbreviated stories in this paper, *Tension* and *We are the Future*. In terms of the IDEA framework, these stories primarily illustrate the *E* and *D* characteristics of the framework. The first narrative, *Tension*, highlights the notion that pure mathematical inquiry is an Exploration of ideas. The second narrative, *We are the Future*, highlights the idea that mathematical practices (particularly standards of proof) are Dynamic, negotiated through Argumentation. The notion that our mathematical ideas are part of our Identity will be explored in-depth in another paper presented at the conference on RUME 2018. I tell the two stories now, followed by discussion and conclusions.

Tension: Pure Mathematical Inquiry is an Emotional Exploration of Ideas

One of the first significant realizations I had during my inquiry into pure mathematics was that engaging with pure mathematics involves an emotional exploration of ideas. One night I began to work on Dr. Combinatorial's conjecture, and I wanted to summarize the important theorems I had just begun to understand. I wished to solidify them in my own mind so that I could make progress on finding a proof for the conjecture. I sat on my bed, writing theorems and proofs in my notebook. Upon writing a proof for a simple result, I noticed a tension. In at least one line, it is clear that I was writing the proof as I would write a proof in my graduate mathematics courses, as if I expected it to be read and graded. I labeled a 7-cycle as $u_1 - u_2 - u_3 - u_4 - u_5 - u_6 - u_7 - u_1$, but I did not use this symbolization elsewhere in the proof. Rather, I convinced myself of the truth of the conjecture through informal methods—drawing a diagram and counting possible chords. I could have written a formal argument, but it did not seem necessary. The tension is that on the one hand, I was working for personal understanding and on the other I was writing with the standards of rigor I believed to be expected in mathematical writing. The conflict is between a personal exploration and understanding of ideas versus the crafting of a communicative proof that satisfies perceived norms of rigor and symbolization.

After proving that theorem I moved onto another one, which involved a proof by induction. I wrote out minute details of the basis step for the n=0 and n=1 cases that were already clear in my own mind (but may not have been clear to a reader). I then wrote, "I find myself realizing this proof is more for me than another. I don't need to communicate all the details. The magic of mathematics is in the ideas one experiences when proving." Essentially I was giving myself permission, with those words, to drop any unnecessary symbolism and tedious explication, and just explore the mathematical ideas (and document that exploration). The very next thing I wrote was, "Out of curiosity, can I show [the n=2 case]?" I already knew a proof by induction could prove for all cases, but I decided to look at a specific case so I could better understand the general argument. I worked through this case myself, drawing several interesting figures. Then I wanted to keep going. I went on to prove the n=3 case. I was enjoying looking at the individual cases, and gaining insight through my work on them. I found the ideas involved in these types of proofs intellectually stimulating. As I began exploring the mathematical ideas related to this conjecture, I found deep satisfaction. Pure mathematics is an enjoyable exploration of ideas. The mathematics came alive through the proving process. Consider this journal entry:

> It is interesting how I see the problem forming. The proof of the problem is different in nature than the class of graphs the proof refers to. The proof has its own concept imagery in my mind—different mathematical processes and procedures disjoint from the class of graphs itself. … The mathematics is alive within the proof. When I imagine the truth of the conjecture, it is some sad lonely objective reality. But the proof is where the magic is. It is where my mind is. It is where the structure can be seen.

We are the Future: Mathematics is Dynamic and Forever Changing

One day near the end of a transition-to-proof class session, students were debating how much detail they needed to put into their proofs. If k is an integer and j is an integer, do you have to write "$k + j$ is an integer" if you use the fact within a proof? And do you have to justify this step by mentioning the closure property of the integers under addition? Some of the students say yes. Others say no. Others want to know if they will be "docked for points" if they do not.

Dr. Amicable says that the students should do whatever the classroom community agrees is best for communication. She asks me what I am thinking and I mention that in professional mathematics papers, there will often be gaps. I say, "It is assumed the mathematician audience knows these things. This sometimes makes the papers difficult for me to read—for someone like me who is not a super mathematician. So I would maybe appreciate some clarity sometimes."

Infinitely Repeating Decimal asks if he, or any other member of the class, were going to write up something for publication, "Would it be viewed in a negative light if it was too expository in areas in which it over explains?" I explain that it is a difference of opinion:

> Surreal: When I wrote my thesis, my professor said, "If we are going to publish this you will have to cut a bunch of stuff." But to me the papers are so hard to read. I would welcome someone coming into the mathematics community who was very explanatory. I just wish more mathematicians could really clearly convey their ideas. But it is just a difference of opinion. There is another mathematician I know who says, "that is the fun of it. You have to go check everything yourself and make sure you do all the side work."

That class laughs about this comment. Another student, Odd, recommends footnotes as a "happy medium" and Infinitely Repeating Decimal agrees. Then Dr. Amicable poses an interesting question taking the discussion to a different place: "You know who the next generation of mathematicians are, right?" There is silence until someone hesitantly says, "us."

Dr. Amicable: Yes! Right? So you are the community. And you will be able to determine those things. What counts as proof is really determined by who is in the community. So that's what's really neat. So if you all go out there and say I'm going to become a mathematician, and I'm going to change this. Just like Surreal. He is going to be right along with you. I want to change it so that it is a little bit easier to understand these arguments. Right?

Infinitely Repeating Decimal: We are going to change the world. I am going to change the entire mathematics community just for you.

Conclusion and Discussion

The purpose of the IDEA framework is to be a list of goals for students' understanding of the nature of pure mathematics. I presented two stories: *Tension,* which touched upon my experience of pure mathematical inquiry as an exploration of ideas, and *We are the Future*, which focused upon a classroom discussion about the dynamic nature of mathematics in regard to standards of proof. Dr. Amicable and I tried to paint a dynamic picture of mathematics for our students. We told them they were the future of the discipline. We taught them that what counts as a proof is negotiated amongst mathematicians, and gave them the opportunity to debate what makes a good proof themselves. We encouraged them to see the value of mistakes in revising their knowledge.

Although students did have the opportunity to reflect on the dynamic nature of proof standards, I was unable to identify a time when students had the opportunity to experience pure mathematical inquiry as an exploration of ideas (as I did during my work on Dr. Combinatorial's conjecture). While Dr. Amicable and I encouraged students to make meaning of statements before proving, perhaps by constructing examples, what was ultimately deemed credit-worthy in the course was a valid deductive proof. I believe students frequently engaged in a syntactical proof production process like that defined by Weber and Alcock (2004):

> We define a syntactic proof production as one which is written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way. […] In the mathematics community, a syntactic proof production can be colloquially defined as a proof in which all one does is 'unwrap the definitions' and 'push symbols'. (p. 210)

In the transition-to-proof course, students' ability to write deductive proofs was prioritized over the ability to explore mathematical ideas. Perhaps it is a sign of the times, a result of the culture. According to Hersh (1997),

> Mathematics as an abstract deductive system is associated with our culture. But people created mathematical ideas long before there were abstract deductive systems. Perhaps mathematical ideas will be here after abstract deductive systems have had their day and passed on. (p. 232)

Are we satisfied to be part of a culture in which students spend less time exploring the ideas behind a theorem than on producing a valid deduction? We must put serious thought into how we structure pure mathematics courses so students develop healthy and productive conceptions of the nature of mathematics. To renew the culture of pure mathematics instruction will require a commitment from instructors and scholars to make choices that promote the values and vision expressed by humanistic philosophers of mathematics, ideas which are represented in the IDEA framework. To bring about changes in students' conceptions of mathematics they must be provided with opportunities to explicitly reflect on their own beliefs about mathematics while also being confronted with positions that challenge those beliefs.

References

Abd-El-Khalick, F., & Lederman, N. G. (2000). The influence of history of science courses on students' views of nature of science. *Journal of Research in Science Teaching, 37*, 1057–1095.

Ball, D. L. (1988). *Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring to teacher education* (Doctoral dissertation, Michigan State University). Retrieved from: http://www-personal.umich.edu/~dball/books/DBall_dissertation.pdf

Beswick, K. (2012). Teachers' beliefs about school mathematics and mathematicians' mathematics and their relationship to practice. *Educational Studies in Mathematics*, *79*, 127-147.

Bishop, A. J. (1988). Mathematics education in its cultural context. *Educational Studies in Mathematics, 19,* 179-191.

Boaler, J. (2016). *Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages and innovative teaching.* Jossey-Bass: San Francisco, CA.

Browder, F. E. (1976). Does pure mathematics have a relation to the sciences? Far from being an esoteric variety of metaphysics, pure mathematics has had and will continue to have a strong and naturally rooted interaction with the sciences. *American Scientist*, *64*, 542-549.

Burton, L. (1999). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics, 37,* 121-143.

Common Core State Standards Initiative (CCSSI). (2010). Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *The Journal of Mathematical Behavior*, *15*, 375-402.

Ernest, P. (1991). *The philosophy of mathematics education.* Bristol, PA: The Falmer Press.

Erlwanger, S. H. (1973). Benny's conception of rules and answers in IPI mathematics. In T. Carpenter, J. Dossey, & J. Koehler (eds.), *Classics in mathematics education research* (pp. 49-58). Reston, VA: NCTM.

Hersh, R. (1997). *What is mathematics, really?* New York, NY: Oxford University Press.

Jankvist, U. T. (2015). Changing students' images of "mathematics as a discipline". *The Journal of Mathematical Behavior*, *38*, 41-56.

Kean, L. L. C. (2012). *The development of an instrument to evaluate teachers' concepts about nature of mathematical knowledge* (Unpublished doctoral dissertation). Illinois Institute of Technology.

Komatsu, K. (2016). A framework for proofs and refutations in school mathematics: Increasing content by deductive guessing. *Educational Studies in Mathematics*, *92*, 147-162.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. New York, NY: Cambridge University Press.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York, NY: Basic Books.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, *27*(1), 29-63.

Larsen, S., & Zandieh, M. (2008). Proofs and refutations in the undergraduate mathematics classroom. *Educational Studies in Mathematics*, *67*, 205-216.

Lederman, N. G., & Lederman, J. S. (2014). Research on teaching and learning of nature of science. *Handbook of Research on Science Education*, *2*, 600-620.

Maciejewski, W. (2016). Instructors' perceptions of their students' conceptions: The case in undergraduate mathematics. *International Journal of Teaching and Learning in Higher Education, 28(1),* 1-8.

Moustakas, C. (1990). *Heuristic research: Design, methodology, and applications*. Newbury Park, CA: Sage Publications, Inc.

Muis, K. R., Trevors, G., Duffy, M., Ranellucci, J., & Foy, M. J. (2016). Testing the TIDE: Examining the nature of students' epistemic beliefs using a multiple methods approach. *The Journal of Experimental Education, 84,* 264-288.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Patton, M. Q. (2015). *Qualitative research & evaluation methods: Integrating theory and practice (4th ed.).* United States: Sage Publications, Inc.

Presmeg, N. G. (2007). The role of culture in teaching. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning,* (pp. 435-458). Charlotte, NC: Information Age Publishing.

Schalk, K. A. (2012). A socioscientific curriculum facilitating the development of distal and proximal NOS conceptualizations. *International Journal of Science Education, 34(1),* 1–24.

Solomon, Y., & Croft, T. (2016). Understanding undergraduate disengagement from mathematics: Addressing alienation. *International Journal of Educational Research, 79,* 267-276.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 127-146). New York, NY: Macmillan.

Tymoczko, T. (Ed.) (1998). *New directions in the philosophy of mathematics: An anthology*. Princeton, NJ: Princeton University Press.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics*, *56*, 209-234.

Weber, K., Inglis, M., & Mejia-Ramos, J. P. (2014). How mathematicians obtain conviction: Implications for mathematics instruction and research on epistemic cognition. *Educational Psychologist*, *49*(1), 36-58.

White-Fredette, K. (2010). Why not philosophy? Problematizing the philosophy of mathematics in a time of curriculum reform. *The Mathematics Educator*, *19*(2), 21-31.

Willoughby, S. D., & Johnson, K. (2017). Epistemic beliefs of non-STEM majors regarding the nature of science: Where they are and what we can do. *American Journal of Physics, 85,* 461-468.