Sparky the Saguaro: A Teaching Experiment Examining a Student's Development of the Concept of Logarithms

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A number of studies have examined students' difficulties in understanding the idea of logarithm and the effectiveness of non-traditional interventions. However, few studies have examined the understandings students develop when completing conceptually oriented exponential and logarithmic lessons that build off prior research and understandings. This study explores one undergraduate precalculus student's understandings of concepts foundational to the idea of logarithm as she works through an exploratory lesson on exponential and logarithmic functions. Over the course of a few weeks, the student participated in a teaching experiment that focused on Sparky – a mystical saguaro that doubled in height every week. The lesson was centered on growth factors and tupling periods in an effort to support the student in developing the understandings necessary to discuss logarithms and logarithmic properties meaningfully. This paper discusses an essential component that students must conceptualize in order to hold a productive meaning for logarithms and logarithmic properties.

Key words: Exponent, Growth factor, Tupling-period, Logarithm, Exponential

The idea of logarithms is useful both in mathematics (e.g., number theory – primes, statistics – regression, chaos theory – fractal dimension, calculus – differential equations) and in modeling real-world relationships (e.g., Richter scale, Decibel scale, population growth, radioactive decay). Therefore, a goal for mathematics educators should be to assist students in developing coherent meanings for the idea of logarithms. How does one achieve this goal? One hypothesis is to research the aspects of the idea of logarithm students have difficulties with. In particular, studies have shown that students have difficulty with logarithmic notation, logarithmic properties and logarithmic functions (Kenney, 2005; Strom, 2006; Weber, 2002; Gol Tabaghi, 2007). Another hypothesis is to develop and test the efficiency of interventions relative to standard curriculum (Weber, 2002; Panagiotou, 2010). Although these methods may shed light on epistemological obstacles students encounter or how successful a non-traditional approach was, neither examine the reasoning abilities needed to coherently understand and utilize the idea of logarithms. In fact, relatively few studies have examined what meanings students have for the idea of logarithms, and fewer have examined how students come to conceptualize the idea of logarithms.

This study investigated one undergraduate precalculus student's understandings of the idea of logarithm and concepts foundational to the idea of logarithm as she worked through an exploratory lesson on exponential and logarithmic functions. The research questions informing this study were:

1. What understandings are foundational to understanding the idea of logarithm?
2. What understandings of logarithmic functions do students develop during an exponential and logarithmic instructional sequence that emphasizes quantitative and covariational reasoning?

The findings of this study revealed an essential component that students must conceptualize in order to hold a productive meaning for the idea of logarithms. That is, in order to reason through tasks involving logarithmic expressions, logarithmic properties, and logarithmic functions in a way that both builds off prior meanings and is useful for more complex tasks, students must first conceptualize that multiplying by A and then by B is equivalent to multiplying by AB. In this study we modeled a student's thinking as she participated in an exponential and logarithmic instructional sequence that included cognitively scaffolded tasks designed to support students in constructing coherent meanings for the idea of logarithm.

Literature Review

Quantitative and Covariational Reasoning

Quantitative reasoning involves conceptualizing measureable attributes of objects and assigning these observations to a quantitative structure (Thompson, 1988, 1990, 1993, 1994, 2011). This way of thinking is critical for developing a coherent understanding of the idea of logarithm. For example, if one conceptualizes $\log_b(x)$ to represent the number of b-tupling periods necessary to x-tuple, then one could reason that $\log_b(b)$, the number of b-tupling periods necessary to b-tuple, should equal 1. The ability to conceptualize the expression $\log_b(b)$ in this way is foundational for their understanding the logarithmic properties and for using logarithms in applied settings. Smith and Thompson (2007) argue that students' ideas and reasoning (with quantities) must become sophisticated enough to warrant the use of algebraic notation and to reason productively with such tools. This investigation was designed to emphasize quantitative reasoning in the context of an exponential situation to motivate students to reason productively with the expressions, equations and functions they define.

The purpose of this study was to uncover the understandings of logarithmic functions students develop when working through an instructional sequence informed by the construct of covariational reasoning. Covariational reasoning is when a student conceptualizes two quantities' values varying in tandem while considering how they are varying together (Thompson & Carlson, 2017). Thompson and Carlson (2017) argue that being able to reason covariationally is crucial for students' mathematical development, especially when constructing meaningful expressions, formulas and graphs. Our lesson begins by attending to two varying quantities individually and then together to influence student thinking as they begin to construct exponential and logarithmic models. Students who are able to reason covariationally may find it easier to coordinate additive changes in one quantity with exponential changes in another quantity (Ellis et al., 2012).

Research Literature on Students' Understandings of Exponents and Exponential Functions

Viewing exponentiation as repeated multiplication is a primitive, yet insufficient interpretation. While some researchers advocate a repeated multiplication approach (e.g. Goldin & Herscovics, 1991; Weber, 2002), others believe this approach limits students (e.g. Ellis, Ozgur, Kulow, Williams & Amidon, 2015; Davis, 2009; Confrey & Smith, 1995). In particular, Confrey and Smith (1995) argue that the standard way of teaching multiplication through repeated addition is inadequate for describing a variety of situations. Weber (2002) proposed that students first understand exponentiation as a process before viewing exponential and logarithmic expressions as results of applying the process. Once this is achieved, the student should be able to generalize the understanding to cases in which the exponent is a non-natural number. Specifically, Weber defined b^x to represent "the number that is the product of x many factors of b" and $\log_b(m)$ to be "the number of factors of b there are in m." If a coherent understanding of exponential functions (and later logarithmic functions) is desired of our students, it is imperative that they have productive meanings for exponents.

Ellis et al. (2015) conducted a small-scale teaching experiment, informed by Smith and Confrey's (Smith, 2003; Smith & Confrey, 1994) covariation approach to functional thinking, with three middle school students that examined continuously covarying quantities. The students were asked to consider a scenario of a cactus named Jactus whose height doubled every week. The authors noticed three significant shifts in the students' thinking over the course of the study: (1) from repeated multiplication to coordinating x and y, (2) from coordinating x and y to coordinated constant ratios, and (3) generalizing to non-natural exponents. The authors noted that a student's ability to coordinate the growth factor (or ratio of height values) with the changes in elapsed time contributed to the student successfully defining the relationship between the elapsed time and Jactus' height. This study leveraged findings from Ellis et al.'s study of Jactus the Cactus to promote more meaningful discussions on logarithms.

Research Literature on Students' Understandings of Logarithms

The topics of logarithmic notation and logarithmic functions often pose a variety of challenges to students (Kenney, 2005; Weber, 2002). Similar to the complexities present in function notation, logarithmic notation consists of multiple parts each with their own dual nature (Kenney, 2005). In the equation $\log_b(x) = y$, b, x, and y take on a variety of meanings (i.e. parameters, variable). Kenney (2005) noted that because function names are often one letter, students do not naturally view log(x) as representing an output to a function. In addition to these unavoidable complexities, Kenney's (2005) study discovered other difficulties students have with understanding logarithmic notation. The data revealed that students displayed mixed understandings of the bases in the expressions. For example, the students appeared to think that different bases always meant the logarithmic expressions were not equivalent (with the inputs being the same). However, when the expression involved the sum of logarithms, some students claimed equivalence because the bases would cancel out. Students also claimed that ln was equivalent to $\log_{10}$. The study also revealed that students would disregard or "cancel out" the word "log" when simplifying equations involving logarithms and solving for x. Despite the aforementioned difficulties, a few of the students were successful in arriving at the correct answer. However, Weber (2002) found that this was an unlikely result of traditionally taught students.

Weber (2002) conducted a pilot study that compared a traditional approach to teaching logarithmic functions with a more conceptual approach that introduced $\log_b(m)$ as the number of factors of b there are in m. Weber's way of discussing the meaning of a logarithmic expression more clearly describes what the multiple parts of the notation represent - therefore addressing the issues Kenney observed in her study. In his study, Weber found that the students who received more conceptually based instruction were more likely to catch their mistakes when it came to identifying and justifying properties of logarithms and exponents. This data emphasizes the importance and need for more coherent and conceptually taught lessons for exponents, logarithmic expressions and logarithmic functions.

Theoretical Perspective and Methodology

The theoretical framework of genetic epistemology (Piaget, 2001) and the theoretical perspective of radical constructivism (Glasersfeld, 1995) form the foundation of this study. A key assertion of radical constructivism is that knowledge is constructed in the mind of an individual and is not directly accessible to anyone else. Steffe and Thompson (2000) label the mathematical constructions made in the mind of a student as “student’s mathematics.” At best, researchers can develop models of student thinking based on the student’s utterances, movements, written work, and essential mistakes. Such models of student’s mathematics are referred to as “mathematics of students” (Steffe & Thompson, 2000). A model is considered reliable when the student acts in a way that remains consistent with the model. The process of developing the mathematics of students is one of scrutiny. Models are formed, tested, revised, and tested again until a viable model is developed. However, to say that a model is reliable is not the same as claiming the model directly represents the student’s thinking – that is an impossible objective. Genetic epistemology focuses on both “what knowledge consists of [cognitive structures - schemes] and the ways in which knowledge develops [what those structures do]” (Piaget, 2001, p. 2). Piaget believed that knowledge is not static, but is always in a stage of development (1977). Therefore, for example, in order to discuss the ways in which students come to understand that $\log_b(m)$ represents the number of b-tupling periods needed to m-tuple, we must develop a model of students’ cognitive structures and a roadmap of what happens to those cognitive structures as students’ knowledge progresses from point A to point B. In this study, we attempt to model the participants’ knowledge development of the ideas foundational to the idea of logarithm.

For this study, we conducted a teaching experiment (Steffe & Thompson, 2000) over the course of a three-week period in an effort to gain insight into student thinking and to develop the mathematics of students regarding logarithms and logarithmic functions. This study consisted of four 1.5-hour sessions with Lexi, a precalculus student, covering the topics of exponential and logarithmic functions in the context of a saguaro cactus that grows exponentially with respect to time (specifically doubling in height each week). Lexi, worked through a packet of questions while referring to a premade Geogebra applet to guide her thinking. As we conducted this teaching experiment, the lesson used was modified as needed during the stages of retrospective analysis. Lexi did not complete any additional assignments between teaching episodes.

Results

This study’s findings identified understandings foundational to the concept of logarithms. The section that follows reports findings that revealed foundational weakness that prevented Lexi from constructing targeted meanings in the lesson. Our findings are supported in our analysis of the discussions between Lexi and me as she completed the tasks.

Foundational Understanding: Multiplying by *A* then multiplying by *B*, has the same effect as multiplying by *AB*

In this section, we present and discuss clips from the teaching episodes that suggest Lexi did not distinguish multiplying by *A,* then multiplying by *B* as having the same effect as multiplying by *AB*. This understanding, or lack thereof, reoccurred throughout the teaching experiment when discussing the meaning of percentages, growth factors and logarithmic ideas. We realized this crucial issue during the retrospective analysis of the third teaching episode and developed a task to allow Lexi an opportunity for reflective abstraction (Piaget, 2001; Thompson, 1985, pg. 196). We conclude this section by discussing the intervention and noting changes in Lexi’s thinking.

The first two episodes focused mainly on percentages, percent change, growth factors and an exponential function. Throughout the first lesson, it became apparent that Lexi had two ways of acting on tasks involving percentages – one more dominant than the next. At first, Lexi associated percentages with a repositioning of the decimal place, but remained in a state of disequilibrium as she proposed a variety of values to represent the percent in decimal form. Lexi resorted to what ended up being her most dominant actions for percent problems. This action entailed Lexi first finding 1% of a value by dividing that value by 100 and then scaling this value to find the desired percent value. For example, to find 73% of $27, Lexi divided the $27 by 100 and took the result, $0.27, and multiplied it by 73 to get $19.71. When Lexi was presented with a percentage task involving multiples of 10%, she acted on the task in a different way. This action involved moving the decimal place of the value she was trying to find the percent of to the left one place (finding 10% of the value) and scaling up to find the multiple of 10. For example, the first author asked Lexi to determine 20% of $27, she moved the decimal place over one place to get $2.7 (10% of $27) and multiplied this value by 2 to get $5.40 (20% of $27).

Although Lexi's dominant action for percentages worked for her, her approach it is not the most productive way to approach tasks involving calculating a percent of a value. To address this observation in the second teaching session, we presented Lexi with the following two questions:

1. Suppose the division button on your calculator wasn't working. How would you determine 1% of $45.67?
2. Suppose the division button on your calculator wasn't working. How would you determine 73% of $45.67?

The purpose of this task was to help Lexi make the abstraction that to determine *n*% of a number, one can multiply by the decimal representation of *n*/100. She began by stating she could divide $45.67 by 100 to calculate 1% of $45.67. We then reminded her that she should assume the division button on the calculator was broken and that she needed to come up with a different way to calculate 1% of $45.67. Lexi's next response was to multiply $45.67 by 1/100. However, we noted that in order to enter 1/100 in the calculator, she would still need to utilize the division button. We followed that statement by asking her, "What is another way to represent 1/100?" and she responded, "0.2? 0.1? 0.01?" – eventually settling on 0.01. When attempting the second problem, Lexi stated, "Don't we just do the same thing?" and said she could determine 73% of $45.67 by multiplying $45.67 by 0.73. Lexi's attention to the results of her actions for the first problem suggests that she developed a new action in her scheme for percentages via a pseudo abstraction (Piaget, 2001). We asked Lexi how she might calculate the same value by using her answer in part (1). She explained that she would just have to multiply the 1% value by 73 to calculate 73% of $45.67. We attempted to draw Lexi's attention to the actions she performed in hopes that she would reflect on her work and abstract that multiplying by 0.73 has the same effect as multiplying by 0.01 and then by 73. That is, multiplying a value by 0.73 finds 73 1/100ths of that value, therefore calculating 73% of the value. Instead, Lexi claimed that the first method uses the 1% and the other (multiplying by 0.73) doesn't "necessarily need the 1% to find (the output)." Lexi's description of the two methods suggests that she viewed them as disjoint from one another. In other words, Lexi's actions suggest she viewed multiplying by 0.01 and then by 73 as being quantitatively different than multiplying by 0.73.

During the remaining portion of the second teaching episode, Lexi worked on a lesson that prompted her to determine different growth factors to represent Sparky the Saguaro's growth. In an attempt to determine the 3-week growth factor, Lexi began by noting Sparky's initial height of one foot at week zero and then claimed, "three time(s)– no, every week it's doubling, or times two for the height. So to get to week three, you'd say it's like, you wouldn't say 6 times as large – that wouldn't make sense. I feel like you would say 3 times as large – that doesn't make sense either." This quote suggests that Lexi first considered multiplying the 1-week growth factor (2) by the number of elapsed weeks (3) to calculate the 3-week growth factor. However, she quickly ruled out that option and looked to other values appearing in the situation. Lexi then appeared to observe the height of the cactus three weeks after its purchase and eventually concluded that the week 3 Sparky would be 8 times as large as the initial Sparky. However, there was no evidence to suggest that Lexi had reflected on the relationship between the 1-week growth factor (2) and the number of weeks that have elapsed (3) relative to the 3-week growth factor (8). In particular, although Lexi noted that Sparky was doubling in height every week, her responses and attention to the heights of the cacti suggest she had not yet abstracted that if Sparky doubles in height three weeks in a row, that will have the same effect as growing by a factor of 2^3, or 8.

During the third lesson, we introduced the biconditional nature between statements involving growth factors and tupling periods. For example, we say the n-unit growth factor is b if and only if the b-tupling period is n-units. In the Sparky context, since the 1-week growth factor is 2, the 2-tupling period is 1 week. Lexi struggled with *n*-tupling periods when *n* was not a power of 2. For example, when we asked Lexi to approximate the 3-tupling period, she claimed it should be 1.5 weeks (so that the three foot Sparky would lie halfway between the 2 foot and 4 foot Sparky). Under the assumption that Sparky was three feet tall after 1.5 weeks, we asked Lexi to determine the number of weeks it would take Sparky to 9-tuple (or to determine the total amount of elapsed time if Sparky 3-tupled in height again). At this point in the teaching experiment, Lexi and the first author had already discussed and concluded that for equal changes in elapsed time, Sparky's height would grow by a constant factor. Therefore, if it took 1.5 weeks to triple, it should take 3 weeks to 9-tuple (but this is impossible since 3 weeks is the 8-tupling period). However, despite our conversations, Lexi's initial response to the 9-tupling question did not appear to rely on her statement that the 3-tupling period was 1.5 weeks. Instead, Lexi claimed the 9-tupling period would be 3.5 weeks and then modified her response to be 3.25 weeks (so that the 9 foot tall Sparky would lie closer to the 8 foot tall Sparky). Again, there was no evidence to suggest that Lexi had reflected on the relationship between the 9-tupling period and the 3-tupling period. In particular, Lexi's response suggests she did not have the understanding that in order to Sparky to 9-tuple in height, he must 3-tuple in height twice. For the remaining portion of the teaching session, Lexi continued to struggle with the idea that if Sparky first *m*-tupled and then *n*-tupled, we could describe his total growth as growing by a factor of *mn*.

After analyzing the third teaching episode and recognizing Lexi's main difficulty, we began the fourth teaching episode with an activity (Figure 1) to allow Lexi opportunities to engage in reflective abstraction on this topic before we introduced logarithmic notation.

(A) At some point in time, Sparky was this tall.	(B) After some time, Sparky 2-tupled in height. Draw the resulting Sparky.	(C) After some more time, Sparky then 4-tupled in height. Draw the resulting Sparky.

Figure 1: Task to address foundational understanding

Lexi drew Sparky (B) and Sparky (C) using a straightedge, documenting the initial height of the intervals and constructing a length that is 2 times as tall and 4 times as tall respectively. Lexi and the first author then had the following discussion:

INT: Sparky (C) is how many times as large as Sparky (A)?

Lexi: Um, wouldn't it be like 6 times as large?
INT: OK, can you verify that?
Lexi: Sure (reaching for straightedge)
INT: And as you are marking that off, can you explain how you concluded it should be 6?
Lexi: Um, well I figured that it would be 6 times as tall because right here this is two times so then that 2 plus that 4 would be 6. (Uses the straightedge to measure how many Sparky (A)'s fit into Sparky (C)) Oh so maybe I was wrong. OK, wait, so it's 8 because is it because it's 4 times 2? Would you multiply those instead of adding them?
INT: Mhmm
Lexi: OK
INT: But can you, can you think about, um, instead of just saying "We're going to multiply instead of add," can you think about why it is multiplication?
Lexi: Um, I guess that would make sense because right here, if you're like doubling it in height, you're multiplying it by two. And then if you're 4-tupling it I guess you are going to increase it by like another factor of 4. So instead of adding the factors you would need to multiply them.

Following this first activity, Lexi correctly completed and interpreted two similar tasks – one where Sparky tripled and then doubled in height, and another where Sparky tripled in height twice in a row. Lexi reasoned with the quantities and was able to conclude that if it took Sparky one week to 2-tuple and approximately 1.58 weeks to 3-tuple, then it should take 1+1.58=2.58 weeks to 6-tuple. In other words, the number of 2-tupling periods (weeks) needed to 2-tuple plus the number of 2-tupling periods (weeks) needed to 3-tuple is equal to the number of 2-tupling periods (weeks) needed to 6-tuple. Symbolically, $\log_2(2)+\log_2(3)=\log_2(6)$ - a specific case of a logarithmic property!

Conclusion

Many studies have examined aspects of logarithms that present difficulties for students, while others have investigated the effectiveness of interventions. In this study, however, we examined the subject's thinking as she participated in a conceptually based lesson on exponential and logarithmic functions. Our findings revealed that the understanding that multiplying by *A* and then multiplying by *B* has the same effect as multiplying by *AB* is crucial throughout a lesson on exponential and logarithmic functions. Types of problems that involve such reasoning include: calculating percentages of values (as witnessed in Lexi's interpretation of finding 73%), determining partial and *n*-unit growth factors (as witnessed in Lexi's struggle with determining the 9-tupling period), representing, interpreting and calculating logarithmic values (in this case, we measure one tupling period using another tupling period), and working with and explaining logarithmic properties (as witnessed with Lexi's interactions in the fourth episode). A student who does not hold this understanding can be successful in answering questions to determine percentages of values, as when Lexi first calculated 1% of a value and then scaled her answer to find a different percent. If our goal is for students to develop coherent understandings of exponential and logarithmic functions, then we must ensure that this foundational understanding is also developed. This finding will be used to improve the Sparky the Saguaro lesson for future research in an effort to provide students more opportunities to develop these foundational understandings at the beginning of the intervention. The Geogebra applet utilized in this study can be requested at egkuper@asu.edu.

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