An Initial Exploration of Students' Reasoning about Combinatorial Proof

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Combinatorial proof involves proving relationships among expressions by arguing that the two expressions count sets with the same cardinality. It is an important topic because it is a kind of proof that has not been studied extensively, yet it represents an aspect of combinatorial reasoning that students should develop. In this paper, we report on data from two students who participated in a paired teaching experiment during which they solved tasks involving combinatorial proof. We highlight some productive aspects of their conceptions of combinatorial proof, and we also report on some pedagogical interventions that seemed to help students progress with successful combinatorial proving. We also argue that combinatorial proofs may naturally tend to be semantic rather than syntactic proof constructions (Weber & Alcock, 2004).

Keywords: Combinatorics, Discrete mathematics, Combinatorial Proof, Proof, Student Thinking

Introduction and Motivation

Binomial identities are equalities that describe relationships between binomial coefficients, such as $\binom{n}{k} = \binom{n}{k-1} + \binom{n-1}{k-1}$. These identities are important because they establish relationships that can be leveraged in a variety of combinatorial settings. While there are often multiple ways to prove such equalities (such as through algebraic equivalences or proofs by induction), a common way to establish binomial identities is through combinatorial proof. Through this technique, we prove that an equality holds by arguing that both sides of the equation count the same set of outcomes. Combinatorial proof tends to be introduced in discrete mathematics or combinatorics classes, and the mathematics community has established the fascinating and valuable nature of this method (e.g., Benjamin & Quinn, 2003). In addition to its use in combinatorial settings, combinatorial proof also provides an interesting setting for students to gain experience with proof and justification. Combinatorics is an accessible mathematical domain, and researchers have made the case that this makes it a useful context for mathematical exploration (Kapur, 1970). Similarly, combinatorial proof could provide an accessible context in which students can gain experience justifying and proving mathematical ideas. In particular, as we will argue, combinatorial proof may naturally provide students experience with semantic (rather than syntactic) proof productions (Weber & Alcock, 2004).

In light of the fact that combinatorial proof is useful for developing both students' combinatorial competency and their proving and justifying, we argue that it is a topic worth studying. However, to date, not much has been explored about this interesting topic. In this paper, we present results from an initial exploration into undergraduate students' reasoning about combinatorial proof. We seek to answer the following research question: *What are key elements of students' conceptions of combinatorial proof that facilitate success with combinatorial proof?*

Literature Review

Literature on Combinatorial Proof

A handful of studies have focused on students' reasoning and activity related to binomial coefficients. For instance, in their longitudinal study, Maher, Powell, and Uptegrove (2011) describe several instances in which students made meaningful connections between binomial coefficients, particular counting problems, and Pascal's Triangle. More specifically, Maher and Speiser (2002) documented student's reasoning about problems involving block towers, which can be solved using binomial coefficients. In a similar vein, Tarlow (2011) reported on eight 11th grade students who could make sense of a well-known binomial identity using both pizza and towers contexts. These studies provide examples of students reasoning about binomial coefficients and identities and show students forming (and in some cases justifying) relationships using combinatorial arguments.

There is another way to think about combinatorial proof, in which each side of an identity counts a different set, and the identity is proved by establishing a bijection between the sets (Mamona-Downs & Downs, 2004; Spira, 2008). The establishment of a bijection is not our emphasis in this study; rather we focus on proofs that count the same set in two different ways.

A Model of Students' Combinatorial Thinking

We draw on Lockwood's (2013) model of students' combinatorial thinking in order to frame our discussion of combinatorial proof; indeed, the model was an integral aspect of the design and analysis of the teaching experiment and design experiment. Lockwood (2013) describes three different components of her model: formulas/expressions, counting processes, and sets of outcomes. Formulas/expressions are the "mathematical expressions that yield some numerical value" (p. 252). Counting processes are "the enumeration process (or series of processes) in which a counter engages (either mentally or physically) as they solve a counting problem. These processes consist of the steps or procedures the counter does, or imagines doing, in order to complete a combinatorial task" (p. 253). Sets of outcomes are "the collection of objects being counted – those sets of elements that one can imagine being generated or enumerated by a counting process" (p. 253). The relationships between these components can help to articulate phenomena that occur when solving counting problems.

To see how this model applies to combinatorial proof, consider the binomial identity $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$, which we call Identity 1. We could establish this identity algebraically by using the definition of $\binom{n}{k} = \frac{n!}{(n-k)!k!}$. However, to prove this identity combinatorially the goal is to demonstrate that both sides of the equation are counting the same set of outcomes. We must first identify the counting process that is represented by each respective expression. Then, we argue that those two counting processes are counting the same set of outcomes. Since that set of outcomes has a certain cardinality, the two expressions will be equal.

For Identity 1, the expression on the left-hand side can be thought of a two-stage process of first selecting a k-person committee from n people, and then selecting m of those k people to be on a subcommittee. Thus, the left counts all possible subcommittees of size m, which were chosen from committees of size k (from a total group of size of n). Alternatively, the two-stage process that reflects the expression on the right-hand side can be thought of as first picking subcommittees of size m from n people, and then picking $k – m$ people from the remaining $n – m$ people to fill out the rest of the subcommittee. The right hand thus also counts the same set, and we can conclude that the identity holds.

In terms of the model, we view this combinatorial proof as being represented by the flow of arrows in Figure 1. Given a relationship between formulas/expressions, we identify two counting processes that reflect the respective formulas/expressions but count the same set of outcomes.

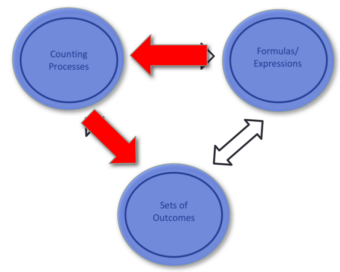

Figure 1. Lockwood's (2013) model of students' combinatorial thinking

We also note that while this direction (formulas/expressions → counting processes → sets of outcomes) reflects how proving combinatorial identities is typically introduced, there are other ways to potentially think about combinatorial proof in terms of the model. In particular, one way to introduce combinatorial identities is through leveraging the fact that there may be more than one way to solve a problem. So, following the example of Identity 1, we could consider trying to answer the question "How many ways can you choose committees of size *k* from *n* people, each of which has a subcommittee of size *m*?" If we tried to solve this problem in two different ways, two natural solutions would be first to pick the committees and then pick the subcommittees, or first to pick the subcommittees and then to pick the committees around them. In this way, we start with the set of outcomes, then we build up two counting processes, ultimately determining two respective formulas/expressions to reflect those processes.

Theoretical Perspective

Harel and Sowder (1998) define *proving* as "the process employed by an individual to remove or create doubts about the truth of an observation" (p. 241). We adopt this definition of proving and consider a proof to be the product of the proving process. Broadly, we consider *combinatorial proving* to be this process of removing doubts about the truth of an observation about a combinatorial relationship, and a combinatorial proof is the product of that process. Specifically, *combinatorial proof* is the result of a certain process of counting the same set of outcomes in two different ways. Thus, to be a combinatorial proof the student must leverage some counting argument in order to establish the relationship. As we have documented above, this involves articulating counting processes that count the same set of outcomes (or, counting the same set of outcomes via two different processes that can be reflected in two expressions). In this paper, the observations that are being proven are always binomial identities.

Weber and Alcock (2004) identified two qualitatively different ways in which someone might produce a correct proof, and we use this distinction as a way of conceptualizing combinatorial proof. They define a *syntactic proof production* as "one which is written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way. In a syntactic proof production, the prover does not make use of diagrams or other intuitive and non-formal representations of mathematical concepts" (p. 210). In contrast, they define a *semantic proof production* to be "a proof of a statement in which the prover uses instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws" (p. 210). We interpret that semantic proof productions describe proof productions in which students meaningfully draw on some instantiation of a mathematical object or idea that may be external from the situation at hand. By emphasizing meaning, they highlight the importance of this instantiation providing some meaning that the symbolic proof normally would not. Although this distinction was introduced in terms of formal proofs (specifically in algebra and analysis context), we argue that these terms could still be a useful lens through which to think about combinatorial proof. We will argue that to prove a binomial identity, a combinatorial proof typically reflects a semantic proof production, whereas an algebraic or inductive proof might naturally be representative of a syntactic proof production.

Methods

Our investigation of combinatorial proof is situated within a broader study investigating generalization in combinatorial contexts. For this paper, we present data from a paired teaching experiment, and we focus on those sessions in which we had students engage in tasks related to combinatorial proof. We conducted a teaching experiment (in the sense of Steffe & Thompson, 2000), during which we interviewed two students over 15 hour-long videotaped sessions. The sessions occurred over approximately 6 weeks during the school year, and the participants were monetarily compensated for their time. We sought students who satisfied three criteria. We wanted them a) to be novice counters, without having formal experience with counting in college, b) to demonstrate a disposition inclined toward problem solving, and c) to be able to articulate their thought process. With these criteria in mind, we chose students based on individual hour-long selection interviews during which they solved counting problems. Two students who fit the criteria (Rose and Sanjeev, pseudonyms) were engineering majors enrolled in a vector calculus class. During the interviews the two students worked together at a chalkboard, and they both regularly contributed to the conversation. The interviewer posed tasks and occasionally asked clarifying questions. We describe the tasks below.

In choosing tasks, we were motivated by the idea that it might be productive to have students first gain experience going from sets of outcomes to counting processes to formulas/expressions by essentially asking students to solve counting problems in two different ways. We also thought that students would benefit from considering a concrete problem (involving specific numerical values) instead of starting with a general statement involving variables. Binomial identities are typically stated as general statements (involving variables like n, k, and r), but we felt it would be useful for students to consider specific instances of those relationships. Due to space, we provide only a partial list of tasks in Table 1.

Table 1. Tasks in the teaching experiment

Activity	**Task**
Starting with a specific problem, solving it in two different ways, then moving toward generalization Formulas/expressions → Counting Processes → Sets of Outcomes	Task 1a: How many 15 person committees are there from 25 people? … Can you solve that in two different ways? Task 1b: What about n people and k people committees? How would you count them in two different ways? Task 2a: There are 10 people, and I want a committee of size six, there is one appointed chairperson. How many such committees are there, and can you solve it in two different ways? Task 2b: Now what if there are n people with committees of size k and a chairperson? Can you solve it in two different ways?
Giving students the binomial	Task 3: $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$

identity and having them argue they count the same set of outcomes. Sets of Outcomes → Counting Processes → Formulas/Expressions	Task 4: $\binom{n}{k} = \binom{n}{k-1} + \binom{n-1}{k-1}$ Task 5: $\sum_{k-0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$

The interview sessions were all transcribed. We created enhanced transcripts, which involved inserting relevant images and gesture descriptions into the transcript. We reviewed the enhanced transcripts of the teaching experiment first and wrote down interesting phenomena about students' reasoning about combinatorial proof. Once we had broad themes we then used the qualitative data analysis software MAXQDA to identify and code relevant data segments. We then synthesized and coordinated our themes into a coherent narrative. We also went through and identified proof productions that we determined to be syntactic or semantic. Due to space, we highlight only a couple of salient findings.

Results – Students' Conceptions of Combinatorial Proof that Facilitate Success

Students Should Understand What It Means to Prove a Relationship "Combinatorially"

Not surprisingly, it was not trivial for students to reason about what was entailed in a combinatorial proof, but we argue that it is important for them to develop this understanding. One noteworthy phenomenon is that the students had to reckon with what a proof is and why counting the same set might actually constitute a mathematical proof.

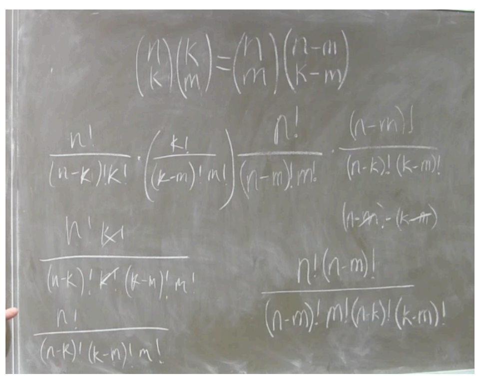

Figure 2. The students establish an algebraic proof

We asked Rose and Sanjeev to prove Task 3 (written at the top of Figure 2), and they asked if they could write it out in a different way. They then immediately started to write out the expressions and work toward an algebraic proof (Figure 2). Although the students were able to make combinatorial arguments, which they had demonstrated by correctly solving previous tasks combinatorially, their instinct was to use an algebraic justification. They subsequently went on to solve the problem combinatorially, but interestingly, even after providing combinatorial arguments, the students seemed more convinced by algebraic arguments.

Interviewer: You proved it algebraically, but suppose you hadn't. Would you be convinced then by your argument that that equation has to be true? Like, are you pretty convinced that that equation is true?

Rose: Uh-huh.

Sanjeev: If we didn't see algebra?
Interviewer: Yep.
Sanjeev: Probably not.
Rose: No.

On Task 4, the exact same phenomenon occurred, where the students immediately tried to prove it algebraically even after they had just combinatorially proved Task 3. This phenomenon is not necessarily surprising. It is important to note that these students were novice provers. As vector calculus students, they had not taken a course involving mathematical proving, and they likely had not been previously confronted with the question of what it means to prove a relationship (they may have seen 2-column geometry proofs in high school, but they had not taken a proof-based undergraduate mathematics course). Thus, it makes sense that perhaps the students' only way of understanding how to establish the equation would be to demonstrate equality through algebra. Nonetheless, even though these students were new provers, we do gain some insight from their work. In particular, their work suggests that when students are introduced to combinatorial proof and combinatorial identities, it may be worthwhile to have a discussion of what it might mean to prove an identity combinatorially.

This data suggests to us that developing a combinatorial proof is understandably nuanced. This implies that students may need to be explicitly taught what combinatorial proof is, both in terms of why it is a valid form of mathematical proof and what is entailed in making a combinatorial argument. Differences between combinatorial versus algebraic arguments might need to be addressed directly if we expect students to understand how to combinatorially prove an identity. Again, this is not surprising, but we have overwhelming evidence that even with very successful and consistent counters, this was a mysterious, new, and challenging idea for them.

Students should develop a particular combinatorial context

Our data also suggests an important aspect of combinatorial proof is for students to be able to reason within a particular context. This is how combinatorial proof tends to be taught, and we are not claiming to offer some new mathematical insight. However, what is noteworthy is that we see evidence of students establishing and leveraging particular contexts, which give them something concrete to count from which they can then generalize. For example, we gave students Task 1 and asked them to count it in two different ways, and their response is seen in the excerpt below (their work is seen in Figure 3).

Interviewer: So are those two things counting the same committees?
Sanjeev: Yeah.
Rose: Yeah.
Sanjeev: The remaining is the number of 15 people committees. In this case, you're making the committees, whereas in that case, you're making them not committees.
Rose: Making them leave.
Interviewer: Okay, but both are giving you the 15 people.
Sanjeev: I think so.
Rose: Yeah, because if you make these ten people leave, then you'll just be stuck with 15 people.

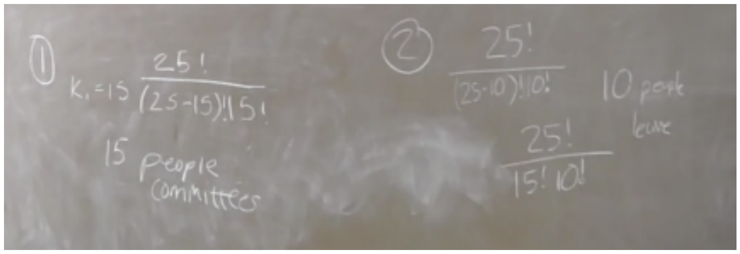

Figure 3. Two different expressions for counting 15-person committees

When we then asked them to generalize we could ask in terms of the same context, and students used the committees context to correctly establish the identity that in Task 1. We contend that being able to reason about and contextualize a problem is instrumental in supporting the combinatorial argument – without it, the formulas/expressions have no combinatorial meaning

Discussion

Combinatorial Proof as Semantic Proof Production

Combinatorial proof is a very specific kind of proof technique. However, although it is narrow, it can also be useful and important for a couple of different reasons. First, it is a specific combinatorial topic that reinforces other important combinatorial ideas like emphasizing sets of outcomes and the relationships between the components of Lockwood's (2013) model. Second, we also argue that it offers a different perspective on mathematical justification and proof. In particular, we propose that combinatorial proof naturally lends itself to semantic proof production. Weber and Alcock (2004) identify several aspects of knowledge required to produce semantic proofs, and we highlight a couple of them as being similar to aspects of knowledge required to produce combinatorial proofs. They emphasize instantiation and say that "One should be able to instantiate relevant mathematical objects. These instantiations should be rich enough that they suggest inferences that one can draw" (p. 229). They also note that "One should be able to connect the formal definition of the concept to the instantiations with which they reason" (p. 229). We interpret that the contextualized combinatorial arguments represent domain-specific instantiations that allow for meaningful proving and justifying. While it may be possible for a student to produce a combinatorial proof that does not involve knowledge that Weber and Alcock describe, we suggest that the kinds of context-based combinatorial justifications required for combinatorial proof are generally indicative of such knowledge.

Conclusion and Implications

Combinatorial proof is a fascinating topic that is relevant both to the teaching and learning of combinatorics and to students' proving activity. In an initial exploration of students' engaging with combinatorial proof we have identified some key conceptions that may help students productively engage in combinatorial proof. We conclude with a couple of potential pedagogical implications of this work. First, students should focus on specific contexts and concrete problems. Then, teachers should give students opportunities to generalize from these particular cases. Our trajectory of concrete to general problems seemed productive in helping students gain familiarity with a context before generalizing using variables. Then, once students have established relationships in the concrete cases, they can attempt the more traditional combinatorial proofs of binomial identities. Overall, teachers should try to make sure students understand how their counting processes relate to the formulas/expressions and the set of outcomes, as discussed in Lockwood's (2013) model.

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