Finite Mathematics Students' Use of Counting Techniques in Probability Applications

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In this study we seek to better understand how students are using counting techniques within the context of the probability application. To do so we investigate three semesters of finite mathematics students' use of enumeration, Venn diagrams, and counting formulas on probability free-response exam questions at a large public university in the mid-south. The study found that appropriate use of enumeration techniques and Venn diagrams both statistically significantly increased a student's likelihood of arriving at a correct answer, while there is statistically significant evidence that the use of counting formulas decreased a student's likelihood of arriving at a correct answer. We conclude with a discussion of the implications of this study for the practice.

Keywords: Combinatorics Education, Probability, Enumeration, Venn Diagrams, Combinations

Introduction and Motivation

The original finite mathematics course debuted in 1957 at Dartmouth College (Kemeny, Snell, & Thompson, 1957). While the course has changed some in the intervening span, it has remained relatively unaltered in the last forty years. This era of stability began when the business major became popular in the United States and business faculty recognized the importance of the finite mathematics course. This change also increased the popularity of the course. The Conference Board of Mathematical Sciences (CBMS) has not yet released their 2015 mathematics programs census reports, the 2010 report gives enough information to conservatively estimate that 120,000 students were enrolled in an introductory finite mathematics course in the United States per annum (Blair, Kirkman, & Maxwell, 2013). Despite having such a long, established history and substantial enrollment, the research literature regarding finite mathematics courses remains limited.

The counting and probability unit taught in Introduction to Finite Mathematics courses is full of versatile topics that matter to the population of students taking the course. This unit has been included in one form or another as a part of the finite mathematics from the courses' beginning in the late 1950s and are not likely to be excluded while business majors are still required to take the course. As Tucker (2013) states, "measuring and counting things [has] interested business-minded Americans from the republic's founding" (p. 692). Despite the interest in these topics and the importance put on them in the CCSSM, they remain largely unstudied at the undergraduate level. Yet, many people utilize counting and probability in their daily decision making without ever being conscious of it.

While it may be that finite mathematics courses and the way instructors teach counting and probability at the undergraduate level are maximally effective, educators cannot know for sure until the topic is fully explored. To date, Elise Lockwood has been the primary contributor to the field. Her studies have been qualitative in nature and have largely focused on students' association of counting with sets (Lockwood, 2011a, 2011b, 2012, 2013, 2014, 2015; Lockwood & Gibson, 2016; Lockwood, Reed, & Caughman, 2016). However, she has not yet addressed the application of counting techniques to probability. Counting and probability are interrelated topics and educators do not know if students are making the necessary connections. The current study allows educators to better understand how students are using counting techniques within the context of the probability application in a finite mathematics course.

As a part of a broader project (Blyman, 2017), this study seeks to address the following research question: *How successfully are students using the counting techniques of enumeration, Venn diagrams, and counting formulas when completing free response probability exam questions?* The results of this study provide insights to improve instruction in courses that include introductory counting and probability.

Literature Review and Framework

All of Lockwood's work has contributed to a better understanding of students' thought processes which can be applied in the classroom. Lockwood (2011a, 2013) posited a model of students' combinatorial thinking where she explored connections students make between counting processes, formulas and expressions, and sets of outcomes. This model paired with mathematical theory relating probability to counting provides a framework for this study.

Lockwood has published various qualitative studies in undergraduate combinatorics education. Throughout her work, she primarily uses student interviews as a tool for gaining insight into the thought processes of students. Investigating counting techniques used by students has led to significant evidence that students struggle to solve counting problems (Lockwood, & Gibson, 2016). More specifically, students "struggle to detect common structures and identify models of underlying problems" (Lockwood, 2011b, p. 307) when solving counting problems. However, the roots of these struggles and ways to mitigate them have not yet been thoroughly studied (Lockwood, 2015).

Particularly relevant to this study are those studies which focus on listing sets of outcomes when working to solve counting problems (Lockwood, 2012, 2014; Lockwood & Gibson, 2016). These studies have resulted in evidence that students understand counting problems best when they enumerate sets of outcomes (Lockwood 2012, 2014; Lockwood & Gibson, 2016). Consequently, Lockwood (2012) encourages students and instructors alike not to be tempted to skip over the crucial step of listing outcomes when learning and teaching students to do counting problems. The multiplication principle connects counting processes with sets of outcomes and, consequently, deserves to be studied in and of itself (Lockwood, Reed, & Caughman, 2016). To begin studying it, Lockwood, Reed, and Caughman (2016) examined many finite mathematics textbooks' treatments of the multiplication principle in counting. They found there were many ways textbooks covered the multiplication principle and hypothesized this could have significant impacts on students' combinatorial thinking (Lockwood, Reed, & Caughman, 2016).

Methodology

Assumptions and Delimitations

This study assumes the course was taught identically by all lecturers and recitation instructors involved as collaboration and sharing of materials was common; however, they were not required to conduct lectures or recitations in the same manner. While informal conversations with lecturers and recitation leaders each semester have made it clear all classes look quite similar, lecturers and recitation leaders each bring their own style and varied experiences into the classroom, so some variance in instruction from section to section likely occurred.

Additionally, data were collected from students enrolled in an introductory finite mathematics course at a large public university in the mid-south for the Spring 2015, Fall 2015, and Spring 2016 semesters. Due to the structure of the course and homework, demographic information was not collected from participants. Consequently, demographics could not be used as covariates in the analyses of this study.

Finally, the format of the exams collected for qualitative data analysis changed between the Spring 2015 and Fall 2015 semesters. The Spring 2015 semester exams consisted of only free response problems while the Fall 2015 and Spring 2016 semester exams each only had 3 free response problems with the rest of the problems being multiple choice. As only the free response questions were considered in this study, the Spring 2015 exams provided much more data for each participant than did the Fall 2015 and Spring 2016 exams.

Data Analysis

To determine how successfully students are using counting techniques on free-response probability exam questions, a stratified random sample of the exams were coded and categorized using the provisional coding method (Saldaña, 2016). Provisional coding makes use of a list of *a priori* codes. For this study the list of codes was created by consulting answer keys to exams which were produced by instructors, considering mathematical connections, and by considering previous research in the field. Since the answer keys to exams from previous semesters are made available online to students as a study tool and the counting techniques which they were exposed to as a part of the course are rather limited, provisional coding using these *a priori* codes was an appropriate choice. While the textbook includes a section on the multiplication principle, the lecturers who wrote the answer keys chose to make use of combination notation over use of the multiplication principle. After scanning several exams and noting how much they resembled the answer keys from previous semester, it was clear that the set of *a priori* codes developed was sufficient to determine how successfully students were using enumeration, Venn diagrams, and counting formulas.

The strata for the sample were formed by recitation leader and the semester the data were collected in order to best form a truly representative sample. For each full-time recitation leader (leading *4* sections) *18* exams were selected and for each half-time recitation leader (leading *2* sections) *9* exams were selected. Sampling was used in order to make the data set more manageable. The numbers *18* and *9* were chosen because it resulted in over one quarter of the exams being coded. In total, *208* exams (*31.3%*) were coded. We scanned over the remaining exams to ensure the stratified random sample was a reasonable representation of all the exams collected. When coding exams, we considered any listing of elements of sets of outcomes to be enumeration, any attempt at using a Venn diagram to be using a Venn diagram, and any attempt at using a counting formula – even simply using combination notation – as making use of a counting formula. The exam answer keys provided examples of what each of the codes represented for reference throughout the coding process. Each part of each free response probability question was treated as an individual problem to be coded. During the coding process, neither coder found any student exams that warranted the addition of a new code.

Both authors coded the exams. Ten exams were coded together to establish consistency. To further guarantee consistency, both intra-rater and inter-rater reliability studies were conducted (Huck, 2012). Each grader reanalyzed a random sample of five exams to measure intra-rater reliability. Intra-rater reliabilities were *86.3%* and *91.5%*. We both analyzed a random sample of ten exams to establish inter-rater reliability. Inter-rater reliability was *86.8%*. In addition to categorizing techniques students were using, the study made use of frequency of codes in a quantitization process (Miles, Huberman, & Saldaña, 2014). Quantitization was used so that it could be determined how successfully students were using each of the counting techniques. Quantitizing the data allowed us to objectively determine if students were answering questions correctly more often or not when they appropriately used enumeration, Venn diagrams, or counting formulas. Using the quantitized data, three chi-square tests of association were conducted pairing each counting technique with correctness of the student's response. Based upon previous studies in the field and by recognizing Venn diagrams as a way of semi-enumerating a larger population by segmenting it, we hypothesized that enumeration and Venn diagrams would be positively associated with correctness on exam problems, while counting formulas would be negatively associated with correctness on exam problems.

Results

Each of the chi-square tests was conducted with the null hypothesis that correctness is not associated with the use, or lack of use, of the given counting technique.

Enumeration

The first of these tests examined the relationship between correctness and enumeration of possible outcomes. The questions included in this test were those on which the instructor chose to utilize enumeration on the published answer key for the exam and those questions which have an easily enumerated set of possible outcomes which the instructor chose not to list on the answer key. Namely, these were Spring 2015 (*n=81* coded exams) questions 1a-c and 7a-b; and Spring 2016 (*n=90* coded exams) questions 14b-c. This yielded a total of $5 \times 81 + 2 \times 90 = 585$ cases where enumeration was an appropriate technique for students to employ. The results of this chi-square test were $\chi^2(1{,}585) = 34.293, p < .001$. Therefore, this result is statistically significant with a small-medium effect size ($\varphi = .242$) since *.1* is considered a small effect and *.3* is considered a medium effect (Cohen, 1988). While students who chose not to enumerate the set of possible outcomes when solving the exam questions were split relatively evenly between those who got the question correct or not, students who used the enumeration strategy were more than twice as likely to get the question correct as incorrect (see Table 1). This result rejects the null hypothesis and confirms the hypothesis that enumeration is positively associated with correctness.

Table 1. Chi-Square Test Results for Enumeration with Correctness.

	No Enumeration	Enumeration	Total
Incorrect	154	78	232
Correct	147	206	353
Total	301	284	585

Venn Diagrams

The second test examined the relationship between correctness and the usage of Venn diagrams. The questions included in this test were those on which the instructor chose to utilize a Venn diagram on the published answer key for the exam. Namely, these were Spring 2015 (*n=81* coded exams) question 5c; Fall 2015 (*n=27* coded exams) question 15a; and Spring 2016 (*n=90* coded exams) question 15a. This yielded a total of $81 + 27 + 90 = 198$ cases where using a Venn diagram was an appropriate technique for students to employ. The results of this chi-square test were $\chi^2(1{,}198) = 5.942, p < .05$. Therefore, this result is statistically significant with a small- medium effect size ($\varphi = .173$) since *.1* is considered a small effect and *.3* is considered a medium effect (Cohen, 1988). While students who chose not to use a Venn diagram when solving the exam questions were approximately two and a half times more likely to get the questions correct as incorrect, students who used Venn diagrams were almost six and a half times as likely to get the question correct as incorrect (see Table 2). This result rejects the null hypothesis and confirms the hypothesis that the use of Venn diagrams is positively associated with correctness.

Venn diagram problems were decidedly easier for students to correctly answer than their enumeration counterparts. Perhaps this is because Venn diagram problems require students to distinguish between at most three distinguishing traits and to classify portions of the population accordingly so there are at most eight numbers which the student is required to determine, while enumeration problems could require students to list as many as *36* possible outcomes to an experiment. On a timed test, students are more likely to not persist and not take the required time to make a list of *36* possible outcomes in such a way as to be able to successfully complete the exam problem.

Table 2. Chi-Square Test Results for Venn Diagram with Correctness.

	No Venn Diagram	Venn Diagram	Total
Incorrect	16	19	35
Correct	41	122	163
Total	57	141	198

Counting Formulas

The final test examined the relationship between correctness and the usage of counting formulas. The questions included in this test were those on which the instructor chose to utilize a counting formula on the published answer key for the exam. Namely, these were Spring 2015 ($n=81$ coded exams) questions 1b, 2a-b, 3a-d, 7a-b, 8a-c, and 9a-b; Fall 2015 ($n=27$ coded exams) questions 13a-c, and 14c; and Spring 2016 ($n=90$ coded exams) questions 13a-c, and 14b-c. This yielded a total of $12 \times 81 + 4 \times 27 + 5 \times 90 = 1692$ cases where using counting formulas was an appropriate technique for students to employ. Students were counted as having used a counting formula if they made any use at all of a formula, even simply using combination notation in their answer. The results of this chi-square test were $\chi^2(1,1692) = 22.636, p < .001$. Therefore, this result is statistically significant with a small-medium effect size ($\varphi = .116$) since *.1* is considered a small effect and *.3* is considered a medium effect (Cohen, 1988). While students who chose to use a counting formula on exam questions were approximately equally likely to get the questions correct or incorrect, students who did not use counting formulas were approximately one and a half times as likely to get the question correct as incorrect (see Table 3). This result rejects the null hypothesis and confirms the hypothesis that counting formulas are negatively associated with correctness.

Table 3. Chi-Square Test Results for Counting Formula with Correctness.

	No Counting Formula	Counting Formula	Total
Incorrect	288	480	768
Correct	453	471	924
Total	741	951	1692

Conclusion

The results of the chi-square tests show students were most successful solving probability problems when using enumeration and Venn diagrams. Students who enumerated sets of outcomes on problems where it was appropriate were more likely to correctly solve the problem than those who chose not to enumerate the set of outcomes on the same problems. Moreover, students who used Venn diagrams on the problems where Venn diagrams were used on the instructor-provided answer key, were much more likely to correctly solve the problem than those who chose not to use a Venn diagram on the same problems. However, students who used counting formulas on the problems where counting formulas were used on the instructor-provided answer key, were more likely to incorrectly solve the problem than those who chose not to use accounting formula on the same problems.

When attempting to solve probability problems set within the context of an inclusion-exclusion counting problem, Venn diagrams were highly effective as a problem-solving technique. A much higher percentage of students correctly responded to the Venn diagram questions than the enumeration or counting formula questions regardless of whether or not a Venn diagram was used. Therefore, students found the Venn diagram problems easier than their enumeration and counting formula counterparts. However, the likelihood of getting a probability question situated in an inclusion-exclusion setting correct increased quite dramatically when a Venn diagram was used.

These results confirm the hypotheses and Lockwood's findings that students understand counting problems best with enumeration (2012, 2014; Lockwood & Gibson, 2016). The chi-square test considering enumeration paired with Lockwood's work make it clear enumeration is an effective strategy for students to use when solving probability problems involving small sets of possible outcomes. Therefore, this study provided quantitative evidence for the findings of Lockwood's previous qualitative studies and extends those findings to the probability application of counting. Additionally, the study provided evidence that students using Venn diagrams have a relatively strong understanding of the problem and are equipped to take steps beyond the construction of their Venn diagram to answer a probability application question related to the Venn diagram that they have created.

Finally, the results of this study make it clear that counting formulas did not help students solve probability problems correctly. In fact, the study revealed students who use counting formulas have a diminished chance of correctly solving the probability problem. At the site of this study, students are offered the option of presenting their final answers as a quotient of two values written in combination notation rather than as the standard proper fraction, decimal, or percentage form of a probability. Through this policy and the presentation of solutions to past exam problems using combination notation, students are not only offered the chance to use this notation, they are strongly encouraged to use it rather than any other counting technique. While, the problems in which students are encouraged to use counting formulas are sometimes more difficult than the problems where they are encouraged to use other counting methods; however, not all the counting formula problems are more difficult. Whether the formula itself was the hindrance or there was some underlying factor at work, between their choice not to explicitly use a counting formula in their work and their misuse of said formula when they chose to use one, the majority of students made it clear they do not understand counting formulas.

Implications for the Practice

The implications of this study for undergraduate mathematics instructors are extensive. These implications stem from the struggles students are having using combination formulas to correctly solve probability problems. Instructors should be encouraging students to use those counting methods which best help them to understand the probability problem which they are attempting to solve. By allowing students to leave their answers in combination notation rather than requiring them to arrive at a proper presentation of a probability, instructors are allowing students to use a counting method without requiring the students know how to use it. Not only are they allowing this phenomenon to occur, they are actually encouraging it by providing students with answer keys to previous semesters' exams in which the solutions are only given in combination notation rather than as a decimal or fraction which the student could use to check an answer arrived at in a different way.

Additionally, since students are largely succeeding at enumeration and are struggling with properly applying the combination formula, it would be advantageous for instructors to give more attention to the multiplication principle – a known intermediary. In fact, given the nature of the counting and probability problems presented at the introduction to finite mathematics level, instructors should be reconsidering if the presentation of combination and permutation formulas is appropriate for the audience. With the limited time allotted for the counting and probability unit, perhaps students would be better served given a conceptual understanding of combinations and permutations while the methods for solving combination and permutation counting and probability problems are restricted to applications of the multiplication principle.

References

Blair, R., Kirkman, E. E., & Maxwell, J. W. (2013). *Statistical abstract of undergraduate programs in the mathematical sciences in the United States: Fall 2010 CBMS survey*. Retrieved from http://www.ams. org/cbms/

Blyman, K. K. (2017). *Influences of Probability Instruction on Undergraduates' Understanding of Counting Processes* (Doctoral dissertation). Retrieved from https://uknowledge.uky.edu/

Cohen, J. (1988). *Statistical power analysis for the behavioral sciences*. Hillsdale, NJ: L. Erlbaum Associates.

Huck, S. W. (2012). *Reading statistics and research*. Boston: Pearson.

Kemeny, J. G., Snell, J. L., & Thompson, G. L. (1957). *Introduction to finite mathematics*. Englewood Cliffs, NJ: Prentice-Hall, Inc.

Lockwood, E., & Gibson, B. R. (2016). Combinatorial tasks and outcome listing: Examining productive listing among undergraduate students. *Educational Studies in Mathematics, 91*(2), 247-270.

Lockwood, E., Reed, Z., & Caughman, J. S. (2016). An analysis of statements of the multiplication principle in combinatorics, discrete, and finite mathematics textbooks. *International Journal of Research in Undergraduate Mathematics Education*, *1*(1), 1-36.

Lockwood, E. (2015). The strategy of solving smaller, similar problems in the context of combinatorial enumeration. *International Journal of Research in Undergraduate Mathematics Education*, *1*(3), 339-362.

Lockwood, E. (2014). A set-oriented perspective on solving counting problems. *For the Learning of Mathematics*, *34*(2), 31-37.

Lockwood, E. (2013). A model of students' combinatorial thinking. *The Journal of Mathematical Behavior*, *32*(2), 251-265.

Lockwood, E. (2012). Counting using sets of outcomes. *Mathematics Teaching in the Middle School*, *18*(3), 132-135.

Lockwood, E. N. (2011a). *Student approaches to combinatorial enumeration: The role of set-oriented thinking* (Unpublished doctoral dissertation). Portland State University, Portland, OR.

Lockwood, E. (2011b). Student connections among counting problems: an exploration using actor-oriented transfer. *Educational Studies in Mathematics*, *78*(3), 307-322.

Miles, M. B., Huberman, A. M., & Saldaña, J. (2014). *Qualitative data analysis: A methods sourcebook*. Thousand Oaks, CA: SAGE Publications.

Saldaña, J. (2016). *The coding manual for qualitative researchers* (3rd ed.). Thousand Oaks, CA: SAGE Publications.

Tucker, A. (2013). The history of the undergraduate program in mathematics in the United States. *The American Mathematical Monthly*, *120*, 689-705.