Analyzing Narratives About Limits Involving Infinity in Calculus Textbooks

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We analyze Calculus textbooks to determine to what extent narratives about limits at infinity and infinite limits align with research in mathematics education. As reasoning about limits falls within the domain of advanced mathematical thinking (AMT), we looked for evidence of appropriate treatment of, and support for, AMT: clear and precise narratives, deductive and rigorous reasoning, intuitive development that does not create or enhance students' misconceptions, opportunities for "personal reconstruction" (Tall, 1991), adequate representations, and the appropriate use of definitions. In conclusion, both high school and university Calculus textbook narratives do not place infinity in a precise, well-defined context, thus possibly creating or strengthening (novice) students' misconceptions. We identified very little evidence of the type of support for AMT that we were looking for. This paper concludes with several suggestions for possible modifications of narratives which involve infinity.

Keywords: Narratives, Limits at Infinity, Infinite Limits, Advanced Mathematical Thinking, Mathematics Textbooks

This study reports on an analysis of presentations of the concept of infinity in textbooks. To be more specific—our aim is to determine if, or to what extent, and how, research in mathematics education has informed, and possibly affected, narratives about infinity in Calculus textbooks. We focus on infinity in the context of limits: limits at infinity (i.e., independent variable "approaches infinity") and infinite limits (i.e., dependent variable "approaches infinity"). This study falls within attempts at bringing research in mathematics education closer to teaching practice.

Discussing ways of improving the quality of mathematics instruction, Artigue (2001) writes: "existing research can greatly help us today, if we make its results accessible to a large audience and make the necessary efforts to better link research and practice" (p. 207). Burkhardt and Schoenfeld (2003) are not optimistic: "In general, education research does not have much credibility—even among its intended clients, teachers and administrators. When they have problems, they rarely turn to research" (p. 3).

In general, mathematics education research rests on well-developed theoretical foundations, and contains constructive information, suggestions and insights for teaching; however, these rarely go far enough and do not touch upon practical aspects of teaching—for instance, by providing content-specific teaching ideas, or by suggesting a rough lesson plan.

Case in point: numerous papers (including almost all cited throughout this paper) address challenges, problems and misconceptions related to teaching and learning infinity at secondary and/or tertiary levels —and yet none gives specific guidelines and suggestions which a textbook writer (or a course instructor) could readily learn from and use. Burkhardt and Schoenfeld (2003) echo this view:

> "The research-based development of tools and processes for use by practitioners, common in other applied fields, is largely missing in education. Such "engineering research" is essential to building strong linkages between research-based insights and improved practice. It will

> also result in a much higher incidence of robust evidence-based recommendations for practice." (p. 3)

There are exceptions. Kajander and Lovric (2009) examine the ways in which the concept of the line tangent to the graph of a function is presented in high school and university textbooks. Their analysis points at exact locations within the narratives that could be problematic (i.e., could lead to the development or strengthening of students' misconceptions), and concludes with specific alternative approaches.

We asked ourselves whether the views presented in Artigue (2001), Burkhardt and Schoenfeld (2003), and others—such as Ball (2000)—accurately portray textbook development of the concept of the limit, in particular when limits involve infinity. The fact that mathematics education researchers—almost as a rule—do not author mathematics textbooks, did not give us much hope that theoretical developments about teaching and learning of limits and infinity found their way into Calculus textbooks.

Limits as Advanced Mathematical Thinking

Tall (1981), Davis and Vinner (1986), Tall (2001), Fischbein (2001), Jones (2015), as well as other researchers (some mentioned in this section, or later in this paper) agree that reasoning about limits falls within the domain of advanced mathematical thinking (AMT). Edwards, Dubinski, and McDonald (2005) write: "AMT is thinking that requires deductive and rigorous reasoning about mathematical notions" (p.17). AMT operates with abstract concepts which require serious mathematical treatment, usually reserved for advanced mathematics courses. Dynamic approaches to introducing the limit, usually discussed in introductory calculus courses, need to be rethought and modified to accommodate for AMT, as otherwise they lead to a variety of misconceptions (Nagle, 2013).

Plaza, Rico, and Ruiz-Hidalgo (2013) assert the importance of definitions as a characteristic that distinguishes elementary from advanced mathematics. Vinner (1991) argues that teaching and learning definitions is a serious problem, and states that a definition "represents, perhaps, more than anything else the conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concept acquisition" (p.65). Edwards and Ward (2004) echo this view: "many students do not *use* definitions the way mathematicians do, *even when the students can correctly state and explain the definitions*" (p. 416).

In Tall (1991), we read: "Advanced mathematics, *by its very nature*, includes concepts which are subtly at variance with naïve experience. Such ideas require an immense personal reconstruction to build the cognitive apparatus to handle them effectively" (p. 252). Edwards, Dubinski, and McDonald (2005) concur, and state that

> "In dealing with limits, students often struggle with the human need to make sense of things by attempting to carry out a process that is impossible to see to the end. Students who view the concept of limit as a dynamic process (meaning a process of getting closer and closer to a limit, but not the object that is the limit) or an unreachable bound, for example, are demonstrating in this instance a failure to use AMT as they are not transcending the finite physical models available to them." (p. 21)

Some researchers identify "process and object components" (Cotrill et al., 1996; Jones, 2015) of the numeric, algorithmic, or theoretical calculation of a limit, with process being equivalent to the notion of "dynamic" in the quote above. For some, "dynamic reasoning" about limits includes both components (Jones, 2015).

Further challenges to creating narratives about limits lie in the language. It is well known that the differences between everyday language and the mathematics language contribute to students' difficulties in understanding (Cornu, 1991; Monaghan, 1991; Kim, Sfard, & Ferrini-Mundy, 2005). Using colloquial phrases such as "to reach," "to exceed," "to approach" in articulating understanding of limits negatively affects students' understanding (Plaza, Ruiz-Hidalgo, & Romero, 2012). Kajander and Lovric (2009) show that this colloquial, "reader-friendly" language leads to the development of misconceptions in textbook presentations of the concept of a tangent line.

Probing narratives can further profit from awareness of the distinction between "transparent" and "opaque" representations in the sense of Lesh, Post, and Behr (1987). As a way of summarizing, Zazkis (2005) writes: "A transparent representation has no more and no less meaning that the represented idea(s) or structure(s). An opaque representation emphasizes some aspects of the ideas or structures and de-emphasizes others" (p. 209).

In conclusion, in our analysis of textbooks, we look for evidence of appropriate treatment of, and support for, AMT: clear and precise narratives, deductive and rigorous reasoning, intuitive development that does not create or enhance students' misconceptions, opportunities for "personal reconstruction" (Tall, 1991), adequate (transparent or opaque) representations, and the appropriate use of definitions.

Infinity in a High School Textbook

We examined one textbook (Dunkley, Carli, & Scoins, 2002), which has been used in grade 12 classrooms in Ontario, Canada. As it accurately reflects the expectations of Ontario high school grade 12 mathematics curriculum (Ontario Ministry of Education, 2007), we believe that this textbook is a likely representative of other textbooks in use.

High school students hear about infinity in a variety of contexts, including: (1) there are infinitely many real numbers; (2) an irrational number is an infinite decimal; (3) limits involving infinity and asymptotes; (4) notation for an infinite interval. (In (1), (2) and (4), we purposely used phrases found in textbooks, to hint at, and to illustrate potential problems.)

Ontario curriculum document (Ontario Ministry of Education, 2007) does not require a clear conceptualization of infinity, for instance by suggesting that infinity be discussed in a *precise, well-defined context.* For instance, the phrase "infinitely *many*" in (1) might suggest a counting approach (*potential infinity*) for a concept that is an instance of *actual infinity*. In (2), it is not clear what "infinite" means—an irrational number has an infinite non-repeating decimal interpretation (i.e., the number is not infinite, but its decimal representation does not terminate). In *Glossary* in Dunkley, Carli, & Scoins (2002), we read that the basis of natural logarithms *e* is a "non-repeating, infinite decimal" (p. 457). Not aware of the subtleties involved, some students think of irrational numbers as infinite (and yet having a finite value). In (4), it is not clear what is infinite about the "infinite interval" (we discuss this further later in this section).

Dunkley, Carli, & Scoins (2002) define infinity as "something that is not finite, that is countable or measurable" (p.459), without explaining what the terms "countable" or "measurable" mean (these two terms do not appear in *Glossary*, nor elsewhere in the text). Not only is this notion confusing, but there is no indication how it relates to the infinity in the context of infinite limits which are discussed in the textbook.

Often, infinity is qualified by what it is not. For example: "We say that the function values approach $+\infty$ (positive infinity) or $-\infty$ (negative infinity). These are not numbers" (Dunkley, Carli, & Scoins, 2002, p. 353). Stating that it is not a number does not clearly articulate what infinity is; the authors continue: "They are symbols that represent the value [plural needed] of a function that increases or decreases without limits" (p.353). Furthermore, the quoted sentences define an infinite limit using the word limit (which makes it a circular definition); or, they are just confusing, as they mix up different meanings of the term "limit" (Jones, 2015). As well, the phrase "without limits" suggests the meaning of the limit as something that bounds, which is a common misconception that students have about limits (Tall, 1991).

Routinely, the same symbol ∞ is used both in interval notation, such as $(1,\infty)$, and in the context of limits. In this case, an adequate narrative needs to resolve this conflict between static and dynamic interpretations of ∞. For instance, treating ∞ in $(1,\infty)$ as transparent (Lesh, Post, & Behr, 1987; Zazkis, 2005), a textbook author could say that the symbol ∞ in $(1,\infty)$ is there for convenience, and could be replaced by some other symbol; all it means ("no more, no less") is that the interval $(1,\infty)$ represents the set of all real numbers larger than 1. We did find such an explanation: Stewart, Davison, and Ferroni (1989) write: "This does not mean that ∞ is a real number. The notation (a,∞) stands for the set of all numbers that are greater than *a*" (p.162).

The use of the term "infinite interval" for intervals such as $(1,\infty)$ is ambiguous, as it is not clear what is infinite about it—the number of points it contains, or its infinite size, or something else? A correct term "unbounded interval," as in advanced calculus/ analysis books, should be used.

Infinity in University Textbooks

Calculus textbooks published in North America since 1980s have been influenced by the so-called calculus reform, or reform-based learning. Besides precise definitions and statements of theorems we find metaphors and explanations which are supposed to help students develop deeper (often intuitive) understanding of concepts. We found out that, in the sections about limits involving infinity, it is these "aids" to building an understanding that are often, due to their authors' disregard for research in math education, worrisome, ineffective and sometimes make no sense. A possible reason for inclusion of narratives (in the case of limits) is an attempt to strike a balance between rigorous (theoretical) development of the limit concept (which is, however, rarely covered in year 1 calculus classrooms) and the need to provide some opportunities for students to deepen their understanding. As well, these narratives could support theoretical understandings, and thus enrich the classroom coverage, often heavily biased toward technical (algorithmic) aspects of limits. Liang (2016) writes: "Calculus teachers usually focus on the calculation of limit, sometimes on graphical illustration of limit, rarely on theoretical aspect (or definition) of limit" (p. 37).

We do not argue against using narratives to enhance understanding, but suggest that they be constructed with care, and with awareness of situations which could lead to the development of students' misconceptions (or complete misunderstandings). In this section we outline a small selection of common narratives that we found in Calculus textbooks. With *novice learners* in mind, we aim to alert instructors to potential issues that these students might face. Of course, certain narratives that we identify as problematic for novices have become an integral part of a language that experts, as well as senior mathematics students, use routinely, and with appropriate and accurate understandings.

All narratives that we discussed in the previous section are found in university textbooks as well. Of the many university Calculus textbooks available to us, we looked deeper into six only, realizing that many have almost identical narratives and identical features. (This important problem of an almost complete absence of variety in presentations, content, and design of Calculus textbooks will not be discussed here.)

Common phrases found in describing an infinite limit of a function "$f(x)$ *becomes* [emphasis added] infinitely large," and "*becomes* [emphasis added] a negative number of large magnitude" (Edwards & Penney, 2008, p. 281) suggest that infinity is "reachable" (an object) as the end of the process of calculating a limit. Instead, a dynamic (process) representation (Cotrill et al., 1996; Jones, 2015) such as "the values of $f(x)$ *surpass* any real number," or similar, should be used.

The phrase "$f(x)$ grows larger and larger" (Jones, 2015) is even more problematic: it suggests that a function has a *size*; however, $f(x)$ is a real number, and has a value, but not a size. The use of "size" as in this example should be avoided; we use "size" when we refer to the set of real numbers, and say that the set of real numbers is infinite (*actual infinity*). As well, "growing larger and larger" does not suffice to guarantee that the limit of a function is ∞.

Another notable feature of textbooks is the absence of a rigorous treatment of infinite limits that would parallel the development of limit laws in the case when all limits involved are real numbers. This absence is truly perplexing, as students are expected to routinely argue about, and use, formulas such as $\infty+1=\infty$, $3\cdot\infty=\infty$, and $\infty+\infty=\infty$. For instance, they need all three of these to compute the limit $\lim\limits_{x\to\infty}(3x+\ln x+1)$. Stretching their intuition further, students might (and do!) erroneously conclude that the indeterminate forms $\infty-\infty$ and $\infty\cdot 0$ are both equal to zero (Lovric, 2012).

More of an exception, we find *Limit Laws for Infinite Limits* explicitly stated in Adler and Lovric (2015, p. 218).

Textbooks often use narratives about infinite limits in the section on L'Hopital's rule. However, almost all that we found are inadequate, or do not contribute to understanding. In Anton, Bivens & Davis (2009), we read:

> "[…] a limit involving $+\infty$ and $-\infty$ is called an indeterminate form of type $\infty-\infty$. Such limits are indeterminate because the two terms exert conflicting influences on the expression; one pushes it in the positive direction and one pushes it in the negative direction." (p. 225)

It is unclear what students are supposed to make of "conflicting influences," i.e., why conflicting influences make an expression indeterminate. For example, in the expression $\lim\limits_{x\to 10}(100x-x^3)$ the two terms $100x$ and x^3 exert conflicting influences (in the sense of the quote above); however, this limit is not an indeterminate form.

In the same book, the authors discuss the indeterminate form 1^∞ coming from the expression $\lim\limits_{x\to 0^+}(1+x)^{1/x}$. They state that 1^∞ is indeterminate because "expressions $1+x$ and $1/x$ exert two conflicting influences: the first approaches 1, which drives the expression toward 1, and the second approaches $+\infty$, which drives the expression toward $+\infty$" (p. 225). Apart from other issues in this narrative, it is completely unclear why approaching 1 and approaching $+\infty$ are "conflicting influences." The two are certainly not conflicting, if we consider $\lim\limits_{x\to 0^+}(1+x)\cdot\dfrac{1}{x}$, i.e., in this case the limit is not an indeterminate form (we replaced exponentiation with multiplication).

Indeterminate forms have also been articulated as "competing forces." When describing ∞/∞, Stewart (2016) uses a somewhat successful metaphor of a "struggle"

> "There is a struggle between numerator and denominator. If the numerator wins, the limit will be ∞ […] if the denominator wins, the limit will be 0. Or, there might be some compromise, in which case the answer might be some finite positive number." (p. 305)

Later in the text, the author uses "contest" (p. 309). However, both metaphors break in the case of exponents, and the textbook offers no explanation as to why 1^∞ is indeterminate.

Hass, Weir, and Thomas (2016) call the indeterminate limit $0/0$ (equivalent to ∞/∞) a "meaningless expression, which we cannot evaluate" (p. 242), without supplying any rationale as to what makes it "meaningless." Hass, Weir and Thomas (2007) use the term "ambiguous expression" (p. 285) when talking about other indeterminate forms; this phrase disappeared from Hass, Weir, and Thomas (2016).

Smith and Minton (2012) are a bit more explicit, when they state that "mathematically meaningless" means that "we'll need to dig deeper to find the value of the limit" (p. 223). However, there is no narrative explaining why these forms are indeterminate (is it just because we need to "dig deeper"?), or suggesting an approach that would help to understand them. Edwards and Penney (2008, p. 296) use inappropriate term "order of magnitude" in discussing the functions in the numerator and the denominator of the indeterminate form ∞/∞ (one possible correct term is "leading behaviour").

Due to space limitations, we presented only a small sample of Calculus textbook narratives about infinity in the context of limits. However, we attempted to select narratives which are more common, and representative of issues and problems that could emerge when students try to read and understand them.

Conclusion

Understanding of, and working with, the concept of the limit requires "an upgrade from intuitive concrete understanding to abstract recognition" (Merenluoto & Lehtinen, 2000). Such an upgrade requires an "immense personal reconstruction" (Tall 1991, p. 252), which includes "deductive and rigorous reasoning" (Edwards, Dubinski, & McDonald, 2005, p.17) and needs to be supported by adequate teaching and resources. Examining a sample of university Calculus textbooks for their treatment of infinite limits and limits at infinity, we have not identified much evidence of this, much needed, support.

Merenluoto & Lehtinen (2000) claim that before students learn about the (mathematical) concept of the limit, they already have experiences about limits. "Their understanding is mainly based on everyday experiences rather than mathematical understandings" (p. 37). As limits are "subtly at variance with naïve experience" (Tall, 1991, p. 252), it is important that textbooks address these experiences head-on to avoid creating or enforcing students' misconceptions about limits. Our examination shows that such narratives are missing from textbooks.

Although all textbooks we examined do cover theoretical aspects of the development of the concept of the limit, they do not dedicate sufficient attention to it. These "theoretical" sections *look* different compared to other sections in the textbooks: with dense presentations, terse language, abundance of symbols and a small number of examples, they seem to have been borrowed from advanced mathematical texts. They are organized in such a way that it is easy for an instructor to skip the material, or to assign it as optional reading. Perhaps we should not be too critical: these "theoretical" sections are an awkward compromise—textbook writers, under pressure from their editors, are forced to include "theory," although they know that many instructors will just skip it. This situation is in line with the general trend of moving theoretical material in Calculus textbooks from dominant to marginal locations (Bokhari & Yushau, 2006).

Nearing the end of this paper, in order to bring our analysis closer to teaching practice, we outline several suggestions, with textbook writers, as well as Calculus instructors, in mind.

Discussing clarity and transparence in teaching infinity, Lovric (2012) writes:

> "We need to make sure that the concepts are precisely *defined*. The necessity for, and a power of a mathematical definition now become obvious. Students will see how the precise and clear language of a definition eliminates multitudes of meanings, inappropriate metaphors and ambiguities in their understanding." (p. 141)

This demands that textbooks, as well as course instructors, bring certain theoretical considerations about limits back to their dominant position. For instance, a precise articulation (definition, together with appropriate illustrations) of the fact that $f(x) \to \infty$ should be accompanied by carefully crafted, transparent, narratives which alert the reader to possible misconceptions and misinterpretations (Monaghan, 1991; Jones, 2015).

All textbooks examined use the phrase "limit does not exist," but mostly do not clearly state that its precise (transparent, "nothing more, nothing less") meaning is "limit is not a real number." As illustration of a possible narrative that attempts to shed some light on this, but is not explicit enough, we quote Smith and Minton (2012): "It is important to note that while the limits […] *do not exist*, we say that they "equal" ∞ and $-\infty$, respectively, only to be specific as to *why* they do not exist" (p.97). Common misconception that "limit does not exist" means that the limit is ∞ or $-\infty$ leads students to conclude, confused, that "infinity does not exist." By the way—besides including infinite limits, "limit does not exist" refers to the case when left and right limits (which could be real numbers) are not equal.

Indeterminate forms should not be referred to as a "meaningless expressions" or "ambiguous expressions." For instance, they could be qualified in the following way: indeterminate forms are algebraic expressions which appear in the context of limits only; they include: division of zero by zero, the cases which are not covered by the limit laws for infinite limits, as well as certain exponential forms involving zero and ∞. (Next, the seven indeterminate forms are listed.) These expressions are called "indeterminate" because their values depend on the limits that generated them; in other words, just by looking at the limits of the form $\infty - \infty$, ∞/∞, or $0 \cdot \infty$, we cannot tell what their values are. For instance, the following four expressions are all of the same indeterminate form ∞/∞, yet the limits are 0, 1, 7, and ∞ respectively:

$$\lim_{x\to\infty}\frac{\ln x}{x}, \quad \lim_{x\to\infty}\frac{x+4}{x-1}, \quad \lim_{x\to\infty}\frac{7x+4}{x+3}, \quad \lim_{x\to\infty}\frac{x^2+1}{x}$$

An introduction to a discussion of infinite limits could start by stating that "'infinity' is really an extrapolation of our finite world, meaning that it is purely a mental construct that we do not encounter in our daily lives" (Tall, 1981, as quoted in Jones, 2015, p. 107). As this *mental construct* appears in various (sometimes incompatible) forms in different contexts, textbooks must be explicit about the context, and then keep their focus. A presentation about infinite limits and limits at infinity (*potential infinity*, *dynamic* nature of infinity) should avoid using terms and phrases such as "size," or "reaches infinity," or any narrative that would suggest objectification of infinity "as a sort of 'generalized large number'"(Tall, 1992, as quoted in Jones, 2015, p. 108).

In conclusion, these are initial, perhaps rough, findings of our analysis. As we probe deeper, we hope to come up with further insights into narratives related to limits involving infinity.

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