Conceptualizing Students' Struggle with Familiar Concepts in a New Mathematical Domain

Igor' Kontorovich
The University of Auckland

*This article is concerned with cognitive aspects of students' struggles in situations in which familiar concepts are reconsidered in a new mathematical domain. Examples of such cross-curricular concepts are divisibility in the domain of integers and in the domain of polynomials, multiplication in the domain of numbers and in the domain of vectors. The article introduces a polysemous approach for structuring students' concept images in these situations. Post-exchanges from an online forum were analyzed for illustrating the potential of the approach for indicating possible sources of students' misconceptions and meta-ways of thinking that might make students aware of their mistakes.*

*Keywords:* concept image, conceptual change, cross-curricular concepts, epistemological obstacles, polysemy

## Introduction

The multidimensional nature of mathematical concepts has been addressed in a number of frameworks. For example, Sfard (1991) and Gray and Tall (1994) distinguished between approaching a concept as a process and as an object. In the former approach, $\sqrt{9}$ is an operation of extracting the square root from the number 9; in the latter approach, it is a number – an object with particular properties. Research suggests that students' fluency with concepts' dimensions and flexibility with switching among them are necessary for developing a deep understanding and for successful problem solving (e.g., Gray & Tall, 1994; Sfard, 1991). Consequently, considerable effort has been invested in supporting students' linkage among concepts' dimensions through stressing their similarities and compatibility (e.g., Moreno & Waldegg, 1991; Sandoval & Possani, 2016). In these studies, the researchers often focused on a particular concept (e.g., infinity, line and vector) and considered them in a singular mathematical domain (e.g., sets, 3-D).

However, in the landscape of students' mathematical education some concepts are reconsidered in different domains. The domains can be rooted in different axiomatic systems and contain different or new objects. Accordingly, a domanial shift of these *cross-curricular* concepts is often accompanied by a substantial change in familiar dimensions (i.e. definitions, properties, procedures, connections with other concepts, etc.). For instance, when extracted in the field of real numbers, $\sqrt{9}$ equals 3; an application of the De Moivre theorem in the field of complex numbers yields 3 and -3 (Kontorovich, 2016a)

The domanial shift and the substantial change in concept dimensions are potential sources for students' difficulties and mistakes (e.g., Kontorovich & Zazkis, 2016). Accordingly, the study reported in this article is concerned with students' struggles with cross-curricular concepts in a new domain. Specifically, my focus is on cognitive aspects of situations in which students, who are relatively fluent with some dimensions of a concept in one domain, encounter its incompatible dimensions in another domain. The aim of the article is to introduce a polysemous approach for analyzing this phenomenon and to illustrate its usage for indicating possible sources of students' mistakes and affordances that might make students aware of them.

## Theoretical Foundations

This section presents the structures of concept image and polysemy, which are then used for presenting the developed approach.

### Concept Image, Terminology and Symbols

The notion of *concept image*, introduced by Tall and Vinner (1981), remains one of the most utilized constructs in mathematics education literature until today (e.g., Panaoura, Michael-Chrysanthou, Gagatsis, & Elia, 2016). The notion refers to the accumulative cognitive structure that a learner associates with the concept, which includes all the mental pictures, properties and processes. Tall and Vinner suggested that it is unlikely that a learner operates with the whole concept image at once and they assumed that various stimuli evoke partial concept images. Accordingly, mathematics education research has been often concerned with exploring tensions among the evoked concept images (e.g., Bingolbali & Monaghan, 2008; Tall & Vinner, 1981).

The terminology and symbols that one associates with a concept can also be considered as a part of her concept image. Extensive research on the tight connections between language and thinking show that discourse shapes our understanding of mathematical concepts (see Austin & Howson, 1979, for an elaborated review). In our case, it seems reasonable to assume that students can identify cross-curricular concepts based on the same terminology and symbols which are used in different domains. This assumption brings up the constructs of homonymy and polysemy.

### Polysemy of Mathematical Concepts

Durkin and Shire (1991) use the notions of *homonymy* and *polysemy* for referring to words with multiple meanings. The meanings of a homonymous word are different and not related, for example "volume" in the sense of a measure of a 3-dimensional object as opposed to intensity of sound. A word is called polysemous if its meanings are related. In terms of this article, it can be proposed that a concept is homonymous if its dimensions in different domains are barely related, for examples a graph of a function in calculus and a graph in graph theory. A polysemous concept, in its turn, can be characterized with dimensions that are valid in different domains and dimensions that hold in particular domains only. Let us take the square root, for instance: The statement "if $b$ is a square root of $a$ then $b^2=a$" is valid for real and complex numbers; however, thinking of $b$ as a non-negative number is appropriate in the former domain only.

A considerable amount of research has been invested in exploring students' understanding of polysemous words with daily and mathematical meanings (e.g., Shire & Durkin, 1989). The polysemy of meanings within the mathematical register is less acknowledged. Zazkis (1998) exemplified the ambiguity of "divisor" with the exercise $12 \div 5 = 2.4$ where in the domain of rational numbers, the number 5 can be addressed as a divisor, since it is defined as the denominator of a fraction. However, if the exercise is considered in the domain of integers, 5 is not a divisor of 12 because there exist no integer that when multiplied by 5 equals 12. Mamolo (2010) focused on the polysemy of symbols '+' and '1'. Her analysis accounted for the changes in the definitions and, consequently, in symbols' meanings in the contexts of modular arithmetic, transfinite mathematics, et cetera. Based on their analyses, Mamolo (2010) and Zazkis (1998) argued that polysemy in mathematics is a potential source of struggle for learners. This study can be considered as an examination of their argument.

## The Study

In terms of IES and NSF (2013) this is an early-stage exploratory research that aims to contribute to core knowledge in education by refining and developing theories for teaching and learning. Thus, the research was approached with the *abduction* methodology (Peirce, 1955). The methodology requires identification of a phenomenon of interest (see Introduction) and gives rise to an initial theory. Then the theory is supported and refined through a purposeful corpus of evidence (Svennevig, 2001). Svennevig (2001) argues that while being a less than certain mode of inference, abduction compensates with a vengeance by providing new ideas and developments. Moreover, the methodology relies on contextual judgements, which are necessary for analysing conceptual development.

Ideas and evidence emerged from a project that involved 25 high-achieving ninth-graders who participated in a linear algebra course (see Kontorovich, 2016b for more details). The course was aimed at preparing for and engaging school students in undergraduate education in parallel with their regular school studies. The course instruction could be described with an often criticized "definition-theorem-proof" structure, which was applied in the topics of polynomials, matrices and vector spaces. When introduced, polynomials and matrices were approached as not being connected to each other, but were later reconsidered as instances of a vector space.

After each lesson the students were provided with a list of problems to solve at home. The solutions were not intended for submission, but variations of some of the problems appeared in a quiz in the following lesson. This led the students to active engagement with course materials and with each other. The students were encouraged to collaborate in a special closed-for-public asynchronous web-forum. Forum post-exchanges were reviewed in a search for evidence of students' identification and struggle with cross-curricular concepts. The two illustrations presented in the article were chosen to highlight various aspects of the developed account.

## Polysemous Concept Images

This section introduces the theoretical account of *polysemous concept image* that was developed in the study. I start with an illustration that stimulated the appearance of the account and continue with another illustration of its various aspects. The illustrations comprise abbreviated post-exchanges between students.

### Divisibility in the domains of polynomials and integers

*Johnny*: [1] Hi guys, I think there is a mistake in question 1d: it asks to show that $q(x)|p(x)$ when $p(x) = 3x^3 - 19x^2 + 38x - 24$ and $q(x) = 6x - 8$. I did the division and got $0.5x^2 - 2.5x + 3$, which has fractions so $q(x)$ can't be a divisor.

*Student 1*: I'm not sure that I got you. Why isn't $6x - 8$ a divisor?

*Johnny*: [2] Think about 3|7, you divide and get $2\frac{1}{3}$, a fraction right? So 3 is not a divisor of 7. Same here.

*Student 2*: What about the question 1c?

*Johnny*: [3] It's ok. You divide $0.5x^2 - 3x - 4$ by $1 - 0.5x$, get $-x + 4$ and everyone is happy.

In Tall and Vinner's (1981) perspective, the illustration sheds light on Johnny's concept image of divisibility (or divisor). His image is an ontologically distinct category containing (at least) two types of conceptions: the ones that regard divisibility in the domain of polynomials and the ones that regard divisibility in the domain of integers. For instance, Johnny's utterances [1] and [3] show that he was aware that the two domains contain different elements (i.e. polynomials and integers) and different division procedures. Accordingly, I propose that Johnny's concept image was compartmentalized into *domain-valid conceptions* – ways of thinking that he perceived as valid in one domain but not in another. Furthermore, Johnny was successful with choosing conceptions from the domain that was intended in problem situations in [1-3].

The connections that Johnny drew between the divisibility in each domain allow proposing that both of them were regulated for him by some common set of conceptions. For example, in both domains Johnny used the '|' symbol for denoting divisibility. The element on the left of '|' was the one by which the element on the right was divided. I use *overarching conceptions* to refer to ways of thinking that can be assigned to one's concept image as a whole. Overarching conceptions make one's concept image *polysemous*: consisting of distinct domains that are connected through conceptions which are valid in each of them (see Figure 1 for a schematic representation of Johnny's concept image).

Johnny's reasoning in the domain of polynomials (see [1] and [3]) can be rephrased as "*r(x)* is a result of dividing *p(x)* by *q(x)*. If *r(x)* contains non-integer coefficients, then *q(x)* is not a divisor of *p(x)*". This reasoning resonates with a variation of conventional definitions of a divisor in the domain of integers: "*r* is a result of dividing *p* by *q*. If *r* is not an integer then *q* is not a divisor of *p*". Accordingly, Johnny's mistake in [1] can be explained with a misclassification of a conception, which is valid in the domain of integers, to the set of overarching conceptions regulating the whole image of the divisibility concept.

Johnny's doubt in [1] in the correctness of the assigned problem suggests that he was convinced in the validity of his misclassified conception. What way of thinking might have helped Johnny to question his solution or maybe even to avoid the mistake? A formal definition of divisibility of real polynomials did not seem to help although he was exposed to it in the classroom[1]. Allow me to address the question with a speculative proposal: in the course of his mathematics learning, Johnny engaged with a variety of concepts and domains. In many cases, if a concept was used to manipulate with elements from the domain, the result was also an element belonging to the same domain (e.g., union, intersection and exclusion of sets is a set; calculations with numbers result in a number; operations with functions create a function). It is very likely that Johnny was not aware that he engaged with instances of closure under operation. We can only wonder whether thinking in terms of "operation with elements from the domain often results in an element belonging to the same domain" would have made a difference for Johnny in the problem situation under discussion. Potentially, this way of thinking might have led him to an observation that his reasoning [1-3] was not consistent: in [3], he operated with polynomials having coefficients of a half and negative half, which indicates that he accepted polynomials with non-integer coefficients as elements of the domain of polynomials. However, in [1] he did not accept such a polynomial as a legitimate result of the division operation between two polynomials, which deviates from the presented way of thinking in terms of closure.

I consider a closure as an instance of a *meta-premise* – a generalized way of thinking, which is conceptualized as valid for various mathematical concepts and domains. Additional examples of meta-premises could be formulated as "a contradiction often indicates a flaw in preceding reasoning", "the same symbol usually denotes the same concept", "if a concept has different

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[1] The standard definition that was provided in the classroom stated: Let $p(x)$ and $q(x)$ be polynomials in $\mathbb{R}[x]$. If there is a polynomial $r(x)$ such that $p(x) = r(x) \cdot q(x)$ then $p(x)$ is said to be divisible by $q(x)$ and we denote $q(x)|p(x)$.

definitions then they are likely to be equivalent”. Note that while these meta-premises are valid in many cases, they are not always valid. Accordingly, Johnny could only notice that his reasoning in [1-3] deviates from the meta-premise but there would be no reason for him to interpret the deviation as a univocal indicator of a mistake. Indeed, there are concepts for which the described closure-under-operation premise does not hold, for example, a scalar product of vectors is not a vector but a number.

The presented post-exchange does not contain evidence of Johnny’s engagement with a meta-premise. Next illustration demonstrates how a student can attend and interpret a deviation of their domain-valid and overarching conceptions from meta-premises.

**Multiplication in the domains of vectors and numbers**

In the following post-exchange, students discussed the problem:

In an isosceles triangle *ABC* (*AB*=*AC*) the medians to the legs are perpendicular (*BB’*⊥*CC’*). Find the value of cos *A* (see Figure 2).

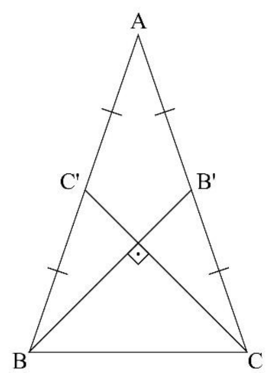


*Figure 2. A triangle from Molly’s problem.*

*Molly*: [1] Please find my mistake: I say that $\overrightarrow{AB} = \vec{u}$ and $\overrightarrow{AC} = \vec{v}$.

[2] So I know that $\overrightarrow{BB'} = \overrightarrow{BA} + \overrightarrow{AB'} = -\vec{u} + 0.5\vec{v}$. In the same way I know that $\overrightarrow{CC'} = -\vec{v} + 0.5\vec{u}$.

[3] Now they are perpendicular so: $\overrightarrow{BB'} \times \overrightarrow{CC'} = 0$.

[4] $(-\vec{u} + 0.5\vec{v})(-\vec{v} + 0.5\vec{u}) = 0$

[5] $\vec{u}\vec{v} - 0.5\vec{u}^2 - 0.5\vec{v}^2 + 0.25\vec{u}\vec{v} = 0$ (actually ignore this part)

[6] $-\vec{u} + 0.5\vec{v} = \vec{0}$ or $-\vec{v} + 0.5\vec{u} = \vec{0}$

[7] and I get $\vec{u} = 0.5\vec{v}$ or $\vec{v} = 0.5\vec{u}$.

[8] But when I come to plug it in the formula I get $\cos A = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{0.5\|\vec{v}\|^2}{0.5\|\vec{v}\|^2} = 1$. So it means that it is a right-angle triangle, which can’t be.

*Student*: What’s the problem with a right-angle triangle? Then you get cos*A=1* and we are done.

*Molly*: [9] I wish… Then we have two different perpendiculars from C to BB’ :(

[10] I probably messed up with vectors.

In terms of the introduced approach, the excerpt shows that Molly holds a polysemous image of the multiplication concept, which she considered in the domains of vectors and numbers. While the former domain was new to her, she demonstrated a high fluency with it: she introduced vectors into a problem (see [1]), added and subtracted them (see [2] and [7]), and manipulated with an inner product for determining angles between vectors (see [3] and [8] where Molly erroneously extracted a right angle from $\cos A = 1$). However, her set of overarching conceptions is a mixture of mathematically correct and invalid concept dimensions: distributivity of multiplication is valid to vectors and numbers indeed (see [4-5]) but the symbols of '×', '·' and an empty space are not tantamount in the vectors domain (see [3-5]). Also, in [6] she presumed that if a multiplication of two vectors equal zero, then one of them is the zero vector. This is another example of a conception that was misclassified from the domain of numbers, in which it is valid, to the set of conceptions overarching the whole image of the multiplication concept.

In contrast to Johnny, Molly was convinced that her solution contained a mistake [8-9]: after determining the measure of angle *A* in the domain of vectors, she reconsidered the obtained triangle with geometry and spotted a contradiction. Accordingly, Molly shifted the assigned problem situation between two domains and interpreted a contradiction in one of them as an indicator of a flawed reasoning employed in another. As a result of engaging with a contradiction, which I consider as an instance of a *meta-premise*, Molly pointed out correctly that she "messed up with vectors" (see [10]). The forum contained no additional data to suggest how Molly continued her work, if at all. However, engaging with a meta-premise helped Molly to become aware of a flaw in her reasoning, which I consider to be a milestone towards restructuring the concept image that she developed.

It is worth mentioning that similarly to the case of Johnny, Molly's engagement with a meta-premise cannot be considered as a dogmatic strategy for verification of developed conceptions and solutions. Molly seemed to take for granted that the concepts of angles, triangles and perpendicularity preserve their dimensions after being shifted from the domain of vectors to the domain of geometry. Clearly this was correct in the particular case. However, spotting an incompatibility of concept's dimensions in axiomatically different domains (e.g., Euclidean and Hyperbolic geometries) requires further work for identifying its source: polysemy of a concept or a flaw in one's concept image and reasoning.

## Summary and Discussion

This article is concerned with cognitive aspects of students' engagement with cross-curricular concepts, and particularly with struggles that can emerge when familiar concept dimensions become invalid in a new mathematical domain. A polysemous approach was introduced for systematizing some sources of students' misconceptions including the affordances that might make students aware of their mistakes. The approach is a theoretical development consisting of overarching conceptions governing one's concept image as a whole and conceptions that are valid in one domain but not in another. A concept image of such a structure was referred in this article as polysemous – fragmentized into compartments which are distinct but related.

The approach was introduced for capturing students' ways of thinking in situations that require a conceptual change. Then, it is not surprising that some aspects of the findings that emerged from data analysis with the approach can be reviewed from the theoretical perspectives of Chi (1992), Vosniadou (2014) and others. For instance, the struggles of Johnny and Molly with separating between domain-valid and overarching conceptions bear resemblance to what Vosniadou (2014) calls "fragmented and synthetic conceptions". These conceptions reflect students' attempts to incorporate new and incompatible knowledge into familiar ways of thinking.

The article contributes to the literature on cognitive change by documenting cases in which students employ meta-premises for indicating the existence of mistakes in their ways of thinking, mistakes that follow from domain-valid and overarching conceptions. Notably, Molly indicated an existence of flaws in her ways of thinking without assigning the flaws to a particular conception. These indications can be associated with an intermediate stage in the process towards a conceptual change, at which learners acknowledge the need for it. Analogously, reviewing a solution or an approach to a problem from the perspective of meta-premises can be interpreted as a metacognitive mechanism that is necessary for carrying out a cognitive change.

An important feature discerning the introduced approach from the literature on a cognitive change is that it acknowledges the existence of epistemological obstacles in mathematics teaching and learning – obstacles that are ingrained in mathematics as a discipline (cf. Brousseau, 1997), polysemy for example. In this way, a polysemous perspective on students' concept images does not only account for the gaps between one's concept image and formal definitions but it also distils the domains in which the developed conceptions are valid. The importance of gaps has been addressed with extensive research conducted with the constructs of concept image and concept definition. Domains of validity are instrumental for recognizing the reasonableness in ways of thinking that students bring to the classroom. Analysis of one's ways of thinking in terms of the approach is aimed at explicating their domains of validity rather than at abandoning them.

Students' struggle with epistemological obstacles of mathematics is unavoidable by definition. In the case of polysemy two features should be considered: First, while progression through mathematical topics can yield opportunities for engaging with cross-curricular concepts, it is still up to the students to identify concepts appearing in different domains as *the same*. Second, fluency with polysemy requires acknowledgment of its existence, experience in making connections between ideas studied in different domains, as well as proficiency in axiomatics and formalism. It is not easy to find educational settings in which a combination of these forms of knowledge is systematically promoted.

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