Cognitive Consistency and Its Relationships to Knowledge of Logical Equivalence and Mathematical Validity

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The purpose of this study is to explore how cognitive consistency is related to knowledge of logic and mathematical proofs. We developed a logic instrument and administered it to forty-seven (47) undergraduate students who enrolled in various sections of a transition-to-proof course. The analysis of the students' scores on the logic instrument indicated that students' knowledge of logical equivalence and their knowledge of mathematical validity were somewhat related to one another. On the other hand, cognitive consistency was not closely related to either student knowledge of logic or knowledge of mathematical validity. Based on these findings, we address the importance of cognitive consistency in logical thinking and discuss implications for the teaching and learning of logic in mathematical contexts.

Keywords: cognitive consistency, logical equivalence, mathematical validity, transition-to-proof

Our society expects people to have ability to make decisions in their workplaces more efficiently by deducing valid inferences from a tremendous amount of information and resources. In fact, a person's logical thinking plays a crucial role in generating valid arguments from the given information as well as in evaluating the validity of others' arguments. Hence, training our students as logical thinkers is a central component in education (NGAC & CCSSO, 2010; NRC, 2005). On the other hand, research in mathematics education reports that undergraduate students have weak knowledge on logic (e.g., Dubinsky, Elterman, & Gong, 1988; Epp, 2003; Inglis & Simpson, 2007). Such a deficiency of knowledge of logic would entail difficulties with using logic in deducing valid inferences to construct mathematical proofs (e.g., Moore, 1994), to comprehend mathematical proofs or interpret mathematical statements (e.g.,Mejia-Ramos et al., 2012; Selden & Selden, 1995), or to evaluate the validity and the soundness of someone's proofs (e.g., Selden & Selden, 2003).

With the importance of student knowledge of logic, we also consider *cognitive consistency* as an essential component in logical thinking. By cognitive consistency, we refer to "an intra-individual psychological pressure to self-organize one's beliefs and identities in a balanced fashion" (Cvencek, Meltzoff, & Kapur, 2014, p.73). Cognitive psychologists explain such a tendency as people behave in ways that maintain cognitive consistency or minimize cognitive dissonance among their interpersonal relations, intrapersonal cognitions, beliefs, feelings, or actions (Festinger, 1957; McGuire, 1966). For instance, a student might deduce two statements such as 'x is an integer' and 'x is not an integer' from given information as well as based on his own content knowledge of mathematics. Logically speaking, each of these statements contradicts one another, thus two statements cannot be accepted simultaneously. Since such a logical contradiction is a fatal flaw that makes the student's entire argument meaningless, it must be excluded from the argument. Once a student recognizes a logical contradiction in his argument, he would attempt to remove it from his argument. However, if the student does not recognize such a contradiction in his argument, he would be in cognitive inconsistency. One's recognition of cognitive inconsistency in his own reasoning or thinking will be the first step in self-regulating one's own cognition. However, if a student does not recognize cognitive inconsistency in his or her own knowledge structures, the student may not take any effort to change or modify his knowledge structure. Thus, it is very important to train students not only to gain more knowledge of logic but also to maintain cognitive consistency.

One might expect that the more knowledge of logic students has, they would unlikely deduce logical contradictions from given information or they would recognize logical contradictions if they happen to deduce them from given information. It might also be expected that students who do not recognize logical contradictions in their arguments would not be knowledgeable in logic. This study explores how students' cognitive consistency is related to their knowledge of logic and knowledge of mathematical validity, addressing the following research questions:

1. Do students with more knowledge of logical equivalence tend to have stronger cognitive consistency?
2. Do students with more knowledge of mathematical validity tend to have stronger cognitive consistency?

We developed the logic instrument to systematically measure three components of students' logical thinking: knowledge of logical equivalence between two statements, knowledge of mathematical validity of the arguments, and cognitive consistency. While we hope that this study provides new insights into the theories of cognitive consistency, our foci are distinct to previous ones from two aspects. First, in exploring the role of cognitive consistency, this study pays more attention to mathematical contexts such as logical equivalence of mathematical statements and mathematical validity, rather than focusing on personal or interpersonal attitudes and behaviors in social contexts (c.f., Cooper, 1998; Gawronski & Strack, 2004; Gawronski, Walther, & Blank, 2005; Stone & Cooper, 2001). Second, this study explores whether students recognize cognitive inconsistencies in their logical thinking rather than how students reconcile cognitive inconsistencies after recognizing them in their reasoning (c.f., Dawkins & Roh, 2016; Ely, 2010; Roh & Lee, 2011).

Research Methodology

This study was conducted in the spring semester of 2014 at a large public university in the United States. Among 137 undergraduate students who enrolled in a transition-to-proof course various instructors, forty-seven (47) students voluntarily participated this study to complete the logic instrument. Due to the pre-requisite for the transition-to-proof course at the university, the participants had already completed at least the first semester calculus course. In addition, as the logic instrument was administered at the last week of the semester when the participants enrolled in the transition-to-proof course, the participants of this study had already been exposed to the terms used in the questions of the logic instrument, such as equivalent statements, logical connectives, quantifiers, negation, and valid arguments. Twenty-three participants (48%) were mathematics majors whereas twelve participants (26%) were mathematics education majors. The rest of the participants (twelve students, 26%) whose major areas of study were neither mathematics nor mathematics education were labeld as others.

The Logic Instrument

The logic instrument we developed for this study consists of two parts with twelve questions in total. The first part (seven questions) was designed to test students' knowledge on logical equivalence between two statements. On the other hand, the second part of the logic instrument (five questions) was designed to test students' knowledge of mathematical validity.

Part 1 of the logic instrument. All questions in Part 1 present one or a pair of statements. We chose logical forms for these questions in Part 1 of the logic instrument among those that are frequently found in undergraduate mathematics textbooks from calculus and beyond. Several instances are also presented with the statement(s) in each question and students are asked to mark off all relevant ones among the given instances (See Table 1). All statements given in the questions in Part 1 of the logic instrument are *open* statements involving at least one free variable so that the truth-value of each statement cannot be determined. We purposely created and included such open statements to the questions in Part 1 in order to avoid the cases of students who answer to the questions based on their determination of the truth-value of a statement.

Table 1 Summary of seven Questions in Part 1 of the logic instrument

	Logical form of the given statements	**Nature of the Questions**
Q1 & Q3	$P(x) \to Q(x)$	Mark off all logically equivalent instances to the given statement
Q2 & Q4	A pair of statements in the forms of $\forall x \exists y P(x, y, z)$ & $\exists y \forall x P(x, y, z)$	Mark off the best description about the logical relationship between the given statements
Q5, Q6, & Q7	$\forall x, P(x, y) \to Q(x, y)$	Mark off all logically equivalent instances to the negation of the given statements

Part 2 of the logic instrument. All five questions in Part 2 are set up similarly in the sense that each question asks to (1) determine the truth-value of the given statement; (2) determine if the person whose argument is given in the question attempts to either prove or disprove the statement; and (3) evaluate if the person's argument is valid. See Figure 1 for Q9 as an example of questions in Part 2 of the logic instrument.

Q9. An integer *a* is said to be odd if and only if there exists $n \in \mathbb{Z}$ such that $a = 2n + 1$. Tim was asked to prove or disprove:

(♣) For any positive integers *x* and *y*, if *x* and *y* are odd, then xy is odd.

The following is Tim's argument.

$$x = 2n + 1, n \in \mathbb{Z}$$
$$y = 2n + 1, n \in \mathbb{Z}$$
Therefore, $xy = (2n + 1)(2n + 1) = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$ is odd.

(1) Check the most appropriate one about the statement (♣).

a. _______ The statement (♣) is true.

b. _______ The statement (♣) is false.

c. _______ We cannot determine if the statement (♣) is true or false.

(2) Check the most appropriate one to describe what Tim attempted to prove.

a. _______ Tim attempted to prove the statement (♣) is true.

b. _______ Tim attempted to prove statement (♣) is false.

c. _______ We cannot determine if Tim attempted to prove the statement (♣) is true or he attempted to prove the statement (♣) is false.

(3) Check the most appropriate one to describe if Tim's argument is valid.

a. _______ Tim's argument is valid as a proof of the statement (♣).

b. _______ Tim's argument is invalid as a proof of the statement (♣).

c. _______ We cannot determine if Tim's argument is valid or invalid.

Figure 1 Q9 in the logic instrument

Data Analysis

The logic instrument described in the previous section was used in this study to measure students' logical thinking in terms of *knowledge of logical equivalence* (KoLE), *knowledge of* *mathematical validity* (KoMV), and *cognitive consistency* (CC). We first generated the coding scheme to score students' mark-offs to the questions in the logic instrument. Different weights were applied to different questions as each question was used to examine different aspects of students' logical thinking. After coding student responses in terms of the scoring rubric, we also generated *the overall logical thinking* (OLT) scores as the sum of the three scores: KoLE, KoMV, and CC scores.

Scoring rubric for knowledge of logical equivalence (KoLE). Student knowledge of logical equivalence was measured from student responses to the questions in Part 1 of the logic instrument (see Table 2). Questions 1, 3, 5, 6 and 7 in Part 1 present a statement and a set of six to seven instances. For each of these questions, sub-question scores were first generated based on students' mark-off to the instances as follows: Students' mark-off to each instance was scored either 0 (for the correct response) or -1 (for the incorrect response). The final score for each of these questions was then formulated as the maximum value between 0 and $2+\sum$(sub-question score). Using this scoring rubric, the scores for Q1, Q3, Q5, Q6, and Q7 were ranged from 0 to 2. On the other hand, Questions 2 and 4 present a pair of statements (i) and (ii) and a set of four instances (a) ~ (d) describing relationships between the pair of statements. For each of these questions, students' check of one of the four relationships was scored either 2 (for the correct response) or 0 (for the incorrect response). KoLE score was then given as the sum of the scores on these seven questions in Part 1, which could be possible ranged from 0 to 14.

Table 2 Scoring rubric for Part 1 of the logic instrument (Q1 ~ Q7)

Question	**Scoring Rubric**			**Score Range**
	Sub-question score		Scoring Formula	Final score
	Correct / Incorrect	score		
Q1, Q3, Q5~Q7	Correct	0	S = max{$2+\sum$(sub-question score), 0}	S
	Incorrect	-1		
	Correct / Incorrect	score		Final score
Q2, Q4	Correct	2		2
	Incorrect	0		0

Scoring rubric for knowledge of mathematical validity (KoMV). Student knowledge of mathematical validity was measured from student responses to the second and third sub-questions to the questions in Part 2 of the logic instrument (see Table 3). First, we evaluated students' student responses to the second sub-question (asking to determine if the given argument is an attempt to prove or an attempt to disprove the statement); and then evaluated student responses to the third sub-question (asking to evaluate the validity of the given argument). To be more specific, for Q8, 1 was given for the correct response to the validity of each argument in the second and third sub-questions, respectively; otherwise 0 was given. For Q9 ~ Q12, 2 was given to the correct mark-off to the second sub-question; otherwise, 0 was given. Next, among those who marked-off correctly to the second sub-question (proof or disproof), if the student also responded correctly to the third sub-question (valid or invalid), we scored 0 for the response to the third sub-question; otherwise, $-\mathbf{1}$ was given. On the other hand, if the student response to the second sub-question (proof/disproof) was incorrect, we scored 0 to any response to the third sub-question regardless of its correctness. We then added the scores on its second and third sub-questions according to the scoring rubric described above. For instance, Q9 (Figure 1) presents an argument (2) attempting to prove the statement is true where (3) the argument is invalid. If a student were to mark off that (2) the given argument in Q9 is an attempt to prove that the statement is false (incorrect), and (3) the given argument is invalid (correct), then 0 was given to this response as the response to the second sub-question is incorrect. On the other hand, if a student were to mark off that (2) the given argument in Q9 is an attempt to prove that the statement is true (correct) and (3) Tim's argument is valid (incorrect), then 1 is given to the student response to Q9 as the correct response to the second sub-question is scored to 2 and an incorrect response to the third sub-question is scored to $-\mathbf{1}$ while the correct response to the first sub-question is neglected due to the incorrect response to the third sub-question. The KoMV score was then given as the sum of the scores on these five questions in Part 2, which could be possibly ranged from 0 to 10.

Table 3 Scoring Rubric for Part 2 of the Logic Instrument (Q8 ~ Q12)

QUESTION	SCORING RUBRIC				SCORE RANGE
	(2) Validity (Argument)		**(3) Validity (Argument)**		**Final score**
	Correct/Incorrect	**score**	**Correct/Incorrect**	**score**	
Q8	Correct	1	Correct	1	2
			Incorrect	0	1
	Incorrect	0	Correct	1	1
			Incorrect	0	0
	(2) Prove/Disprove (Argument)		**(3) Validity (Argument)**		**Final score**
	Correct/Incorrect	**score**	**Correct/Incorrect**	**score**	
Q9~Q12	Correct	2	Correct	0	2
			Incorrect	-1	1
	Incorrect	0	-	-	0

Scoring rubric for cognitive consistency (CC). For cognitive consistency scores, we first identified cognitive inconsistencies when student responses to sub-questions of a question imply any logical contradiction. For instance, suppose a student marks off to Q9 (Figure 1) as follows: (2) Tim's argument is an attempt to prove the statement (♣) is false, and (3) Tim's argument is valid. This student's responses contain a logical contradiction since an attempt to prove that a true statement is false cannot be valid. Similarly, if another student responds to Q9 that (2) Tim's argument is an attempt to prove the statement (♣) is true, and (3) Tim's argument is valid, then the student also appears to have cognitive inconsistency. Table 4 describes all instances of cognitive inconsistencies to be evidently found from student responses to the questions.

Table 4 All instances of cognitive inconsistency

Question		**Sub-Questions**		
Q8	Cognitive Inconsistency	**(1) True/False (Statement)**	**(2) Validity (Argument)**	**(3) Validity (Argument)**
		(a) True or (c) Cannot determine	(a) Valid as a proof for false	-
		(b) False or (c) Cannot determine	-	(a) Valid as a proof for true
Q9~Q12	Cognitive Inconsistency	**(1) True/False (Statement)**	**(2) Prove/Disprove (Argument)**	**(3) Validity (Argument)**
		(a) True or (c) Cannot determine	(b) Prove False	(a) Valid
		(b) False or (c) Cannot determine	(c) Prove True	(a) Valid

Cognitive consistency *was* measured from student responses to all three sub-questions of the questions in Part 2 of the logic instrument. We measured students' cognitive consistency by assigning either -1 or 0 to each of the questions (Q8~Q12) as follows: -1 was assigned whenever there is evidence of cognitive inconsistency, i.e., a logical contradiction from student responses to its sub-questions. On the other hand, we scored 0 in all other cases but the instances in Table 4 since there is no evidence of logical contradictions from the cases. As there were five questions in Part 2, the total score on cognitive consistency could be possibly ranged from -5 to 0. Obviously, if a student marks off correctly to all sub-questions to a question in Part 2, the student does not appear to have a cognitive inconsistency in his response to the question. On the other hand, although the student responses to some sub-questions are not correct, the student's cognitive consistency score to the question could still be 0 in the case when there is no evidence of logical contradiction within the student's responses.

Results

The overall logical thinking (OLT) scores were distributed between -2 and 24 with the interquartile range between 6 and 15. In addition, the mean of the OLT scores was about 10 and the highest OLT score was 24 which was the possible maximum for the OLT score with only one student receiving the highest score. On the other hand, there was one student who received -2 on the OLT scores due to negative values on the cognitive consistency score, which will be discussed later more in detail when analyzing the cognitive consistency scores. Furthermore, KoLE scores were ranged from 0 to 14 while the median was 5 (out of 14 points) and 50% of student KoLE scores were between 2 and 9. KoMV scores were ranged from 0 to 10 with the median 5 (out of 10 points) while 50 % of KoMV scores were distributed between 3 and 8. Finally, the CC scores were ranged from -2 to 0 and about 21% of the CC scores were negative.

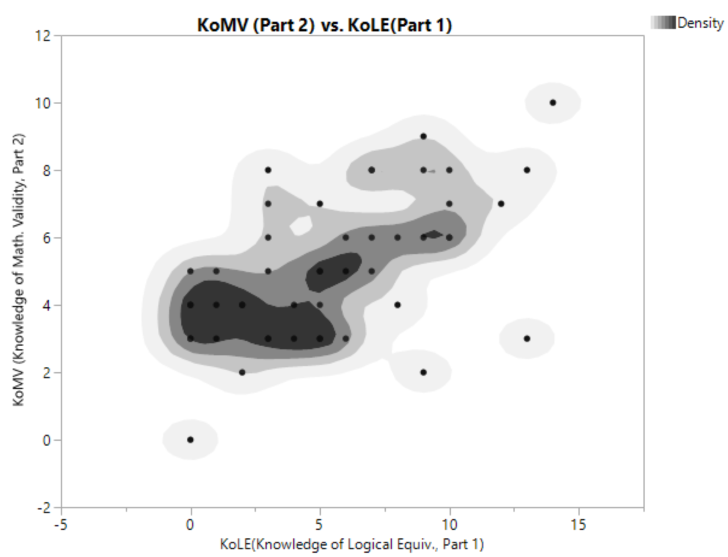

Figure 2 Scatter-Density plot: KoMV vs. KoLE

The scatter-density plot in Figure 2 further shows that students' knowledge of logical equivalence (KoLE) and students' knowledge of mathematical validity (KoMV) are somewhat related to one another. On the other hand, cognitive consistency (CC) was *not* closely related to either KoLE or KoMV. According to the scatter-density plots in Figure 3 and Figure 4, students whose cognitive consistency score was -2 did not have higher scores on KoLE and KoMV than the median of each score. However, in the case that the cognitive consistency score was -1, students’ KoLE scores or KoMV scores were distributed with relatively wide range containing higher scores than the median. There was one student who received a very high score on KoLE (13 out of 14) but scored -1 on the cognitive consistency. These findings indicate that students might have cognitive inconsistencies even though they attained high scores on knowledge of logical equivalence and knowledge of mathematical validity, respectively.

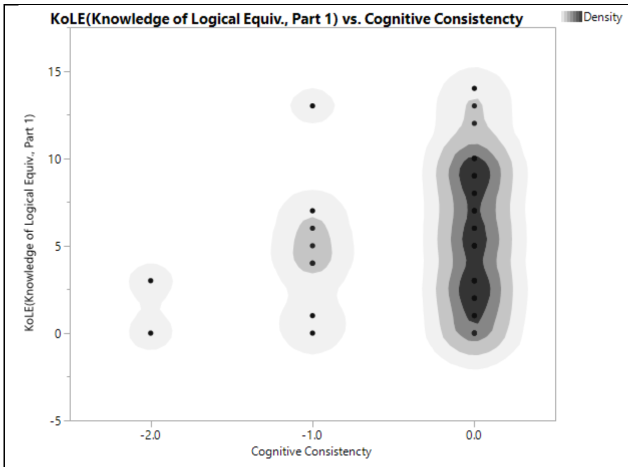

Figure 3 Scatter-Density plots: KoLE vs. CC

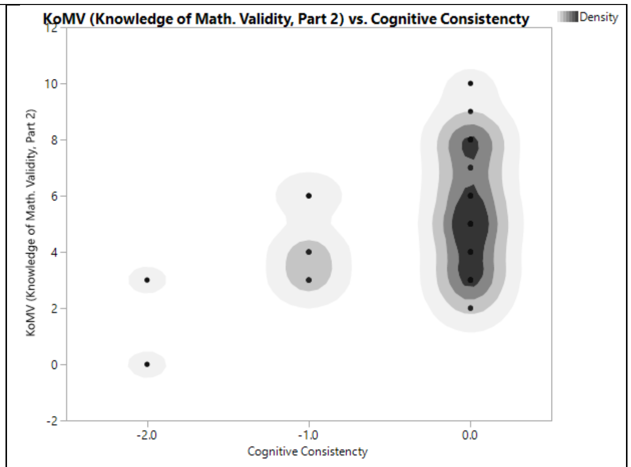

Figure 4 Scatter-Density plots: KoMV vs. CC

Conclusion & Discussion

In this study, we explored undergraduate students’ cognitive consistency and its relation to their knowledge of logical equivalence and mathematical validity. The findings of this study indicate that students’ cognitive consistency was not closely related to either their knowledge of logical equivalence or their knowledge of mathematical validity. Indeed, some students who received high scores on knowledge of logical equivalence or on knowledge of mathematical validity still had cognitive inconsistencies. Furthermore, these students already took a course for logic and mathematical proofs for about at least fifteen weeks. Thus, it might be an unreasonable expectation that students with more knowledge on logical equivalence and mathematical validity would not have cognitive inconsistencies.

The findings of this study also suggest some significant implications for the teaching and learning of logic and mathematical proofs. Although undergraduate students received formal instruction for logic from a logic and mathematical proof course, they may not recognize a logical contradiction in his or her argument. Thus, we contend that cognitive consistency must be treated as a crucial component of logical thinking. Designing special tasks or instructional interventions would be needed to reveal students’ cognitive inconsistencies and to help students recognize logical contradiction in their arguments if they have any. The structure of sub-questions in Part 2 of the logic instrument in this study could be an example of reference to reveal students' cognitive inconsistency what might have been.

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