Stepping Through the Proof Door:
Undergraduates' Experience One Year After an Introduction to Proof Course

Younggon Bae, Michigan State University
John P. Smith III, Michigan State University
Mariana Levin, Western Michigan University
V. Rani Satyam, Michigan State University
Kevin Voogt, Michigan State University

Navigating the transition from computing to proof writing remains a key challenge for mathematics departments and undergraduate students. Numerous departments have developed courses to introduce students to the nature of proof and effective argument (David & Zazkis, 2017), but research assessing the impact of these courses has just begun. This paper reports the experience of four introduction to proof "graduates" after they completed a semester of real analysis. Each had participated in our prior study of students' experience in the introduction to proof course. Results indicate that students' success in real analysis was supported by their work in the introduction to proof course. Two students exploited the structure common to many proofs in real analysis; the other two relied on extensive practice with example problems. For both pairs, we see linkages between students' work in real analysis and their prior procedurally-oriented work in mathematics.

Keywords: transition to proof, proof reasoning, students' experience, qualitative analysis

This paper extends our prior research that examined undergraduate students' experience in one introduction to proof course taught at a research-intensive university (Smith, Levin, Bae, Satyam, & Voogt, 2017). Most of the N = 14 participants in that study clearly indicated that they found the work to write proofs different from their prior work to compute numerical or symbolic "answers". Where the majority found proof writing challenging, most were relatively successful in the course, as judged by final grades and self-reports. But the success of courses designed to introduce students to proof and proof-writing cannot be judged "locally". As the warrant for such courses is to increase learning and achievement in upper-division mathematics, the "success" of these courses depends on how well students perform in subsequent proof-focused courses.

Here we report on the experience of four "graduates" of an introduction to proof course in their first semester of real analysis. All were successful in that course, as judged by both grades and self-reports. But their descriptions of their work in real analysis, offered in comparison to the introduction to proof course and prior work in mathematics through calculus, reveal a more complex pattern of similarities and differences in how students see and carry out mathematical work. For some students in real analysis, the differences between following procedures to compute answers and writing effective proofs may be less stark than we initially conjectured (and than they experienced in their introduction to proof course). If so, characterizing the transition to proof may need to embrace important continuities as well as discontinuities with prior mathematical work.

The Transition to Proof and Proving

Understanding the challenges that undergraduate students face in learning to prove mathematical statements and designing courses and experiences that support their efforts to address those challenges have become major foci of research in undergraduate mathematics education. Recent work has focused on the nature and diversity of courses intended to introduce students to proof and proving (David & Zazkis, 2017; Selden, 2012), specific cognitive challenges in understanding and writing proofs (Sellers, Roh, David, & D'Amours, 2017), and following students' proving work and reasoning after an initial introduction to proof (Benkhalti, Selden, & Selden, 2017).

As one contribution to this growing literature, we interviewed N = 14 undergraduate students after they completed a one-semester introduction to proof course. Our analysis focused on four issues—how they saw the course as different from prior courses, the activities they undertook to learn the course content, how they characterized their thinking during work on proof (proof reasoning), and their sense of success in the course (Smith et al., 2017). None reported any significant prior work on proof. Most were clear that the course made new and different demands than prior courses, and in response, many initiated different patterns of work. Despite their reported struggle, most completed the task being and feeling relatively successful, leading us to conclude that the course had been successful in bringing students to and through the doorway to proof. In particular, the course had placed students into the work of solving mathematical problems and supported their adjustment to that work.

But the merit and impact of introduction to proof courses lies as much, if not more in how students perform after they complete such courses as it does in how well they perform in the courses. These courses typically survey the major domains of algebra, analysis, and number theory without exploring the content area in any depth (David & Zazkis, 2017). The main task of repeatedly addressing proof tasks in one content area for an extended period and thereby coming to understand more about that mathematics via proof lies ahead of them. If the gap between carrying out known procedures to compute single answers and proving statements is wide and deep (Selden & Selden, 2013), the transition to proof and proving will not be accomplished in a single semester. So it makes sense to ask about successful "graduates" of introduction to proof courses: Where does reasonable success in that course lead? How do they experience their first proof-based course situated in a particular content area? How do they compare their experience in that course to their "preparation" in the introduction to proof course and to their prior mathematical experience? Is it possible, at a reasonable level of precision, to chart students' experience and work from computing to proving?

Conceptual Framework

Our analysis was informed by the main concepts that had oriented our prior work (Smith et al., 2017). Oriented by work to understand pre-college and college students' experience of work in "reform" and "traditional" courses (e.g., Smith & Star, 2007), we see the shift from computing single answers to proving statements to set the stage for major transitions in students' experience of mathematics, where their understanding of the nature of their work, how they feel about their experience and their abilities, and what they do to carry out that work change in quite substantial ways. Where mathematical transitions are not determined by the learning environment, some "external" structures make them more likely (e.g., fundamental changes in curriculum and pedagogy and new courses focused on proof). Our prior study conceptualized students' experiences in terms of (a) the differences they saw between the work in their introduction to proof course and their prior mathematics, (b) their sense of the task of writing proofs, (c) their learning activity, in and outside of class, and (d) their subjective "sense of success" in the course. As specified in these four foci, our task was to understand the participants' perspectives and judgments in their own terms.

In the present study, we focused on students' experience in real analysis in relation to their work in the introduction to proof class a year earlier. The above four foci again informed the development of our interview questions and the direction of our analysis. For the first focus (differences with prior courses), we were particularly interested in how students compared the introduction to proof course to real analysis. Our overarching theoretical stance remains constructivist: Students bring forward mathematical "resources" (knowledge, skills, learning practices) developed in prior courses and attempt to use them to address the tasks of their present courses. New challenges, at any scale, mean that some resources will work well, some must be adapted, and some are developed, more or less *de novo*, in the new setting. The view of the student as an agent in her own learning is also central to our perspective, especially with respect to learning activities outside of class.

The Program, Courses, and Participants

In the university where our research is situated, students—both mathematics majors and minors—complete a calculus sequence, the introduction to proof course, a linear algebra course, and a number of proof-focused content courses. After linear algebra, the first semester of real analysis and the first semester of abstract algebra are the two most common sites where students experience proof-intensive work in a specific content area. Both courses are required for majors and minors, and both first require completion of the introduction to proof course. Majors are required to complete additional proof-based courses, including the second semesters of both real analysis and abstract algebra, as well as other courses. In the semester of our study, two sections of real analysis were taught by different instructors, but both used the same textbook (Ross, *Elementary analysis: The theory of calculus*, 2013). They differed somewhat in in-class activities, homework, and assessments. In this department, real analysis is widely seen by students, instructors, and support staff as among the most, if not the most challenging undergraduate course.

In Spring 2017, six of the 14 students from our previous study responded to our invitation to participate in a follow-up study. All the six were mathematics majors or minors and had taken real analysis and/or abstract algebra in 2016-17. The other eight initial participants either did not respond or indicated they had not taken either course, had changed majors, or left the university. Two respondents took both real analysis and abstract algebra; the other four took only one. For those who had taken both courses, our interview focused on the most recent course to reduce concerns about constructed memory. With four participants, the interview focused on real analysis; with the other two, it focused on abstract algebra. In this presentation, we will focus on the former group, who are described in Table 1 below.

Table 1. Overview of participants

Student	Gender	Standing	Home	Major	Career Obj.	Other proof-based courses
S1	Female	3	US	Mathematics	Teaching	Higher geometry (F16)
S2	Female	3	US	Mathematics	Actuary	None
S3	Female	3	US	Mathematics	Uncertain	None
S4	Male	4	Int.	Mathematics	Grad school	Abstract algebra I (F16) Abstract algebra II (Sp17)

All four participants took the course in the same semester (Spring 2017), and we interviewed them just after they completed it. S1, S2, and S3 had the same instructor; S4 was taught by the other instructor. Though we did not directly observe either instructor's teaching as we had in the previous study, S1, S2, and S3 provided consistent descriptions of the course, their instructor's teaching, the assigned homework, the use of the text, and the course assessments.

The interviews were semi-structured around focal questions, about an hour in duration, and conducted either face-to-face or via video conference. In two cases (S1 and S3), follow-up interviews were used to clarify their responses from the first interview. Our central goal was to understand students' experience in real analysis relative to their experience in the introduction to proof course—with particular attention to the task of writing effective proofs. After checking basic information (e.g., major/minor, standing, career plan, other math courses), we asked about their sense of how well the introduction to proof course prepared them for the real analysis course (and any other proof-based courses they had taken). Making no assumptions about how participants saw the mathematics courses they took that year (e.g., linear algebra), we asked how they viewed each relative to its focus on proof (very little, somewhat, strongly). All four participants indicated that real analysis was strongly proof-based. For the course(s) characterized as somewhat or strongly proof-based, we asked participants to compare the difficulty of that course(s) to the introduction to proof course. Then we explored their experience in each course, but with greater attention to real analysis, including assignments and instruction, learning activities in and outside of class, and their view of proof tasks and work to produce acceptable proofs. The interviews also provided opportunities to return to participants' experience in the introduction to proof course, affording us the chance to check for consistency in their characterizations. Toward the end, we asked them to draw a Confidence Graph to represent the dynamics of their confidence across the semester (Smith et al., 2017). As before, these helped us understand the challenges participants faced at different points in the semester and how they addressed the challenges.

Figure 1 below represents the comparisons between the different mathematical experience that were supported in the two studies, the present (Phase 2) and the previous (Phase 1). The interviews from the present study supported comparisons between real analysis and the introduction to proof course, but also with participants' experience prior to any focus on proof.

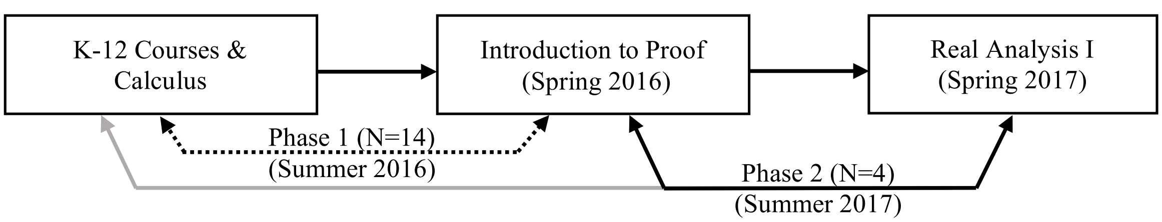

Figure 1. The previous study and the present study across the sequence of the courses

Results

All four students reported success in real analysis, with both final grades (all received 4.0) and sense of mastery of the content. As indicated in Table 2 below, all four appreciated and valued the preparation for real analysis they received in the introduction to proof course, citing (a) "getting their feet wet" with proof, (b) learning specific proof methods (e.g., mathematical induction), and (c) gaining some introduction to real analysis content. However, S1 and S3 made a stronger case for their preparation in the introduction to proof course, where S2 and S4 indicated they did not learn some things that would have been useful in real analysis. S2 stated that she was not required and taught how to build up the structure of a proof; S4 mentioned that some methods (e.g., epsilon-delta proofs) were not taught in detail enough in the introduction to proof course. All four participants noted that the introduction to proof course moved frequently between content areas (making the course more difficult in the process), where real analysis focused on one set of related ideas. S1, S2, and S3 each indicated that they appreciated learning in real analysis why theorems and rules they learned in calculus were justified.

Table 2. Summary of students' sense of preparation of the introduction to proof course for Real Analysis I

Student	Preparation	Relative Difficulty
S1	Very well	Introduction to proof > Real analysis
S2	Good	Real analysis > Introduction to proof
S3	Very well	Introduction to proof > Real analysis
S4	Good	Real analysis > Introduction to proof

S1 and S3 found the introduction to proof course more difficult than real analysis, despite the fact that prior reports led both to expect that the latter would be very challenging. In contrast, S2 and S4 indicated that real analysis was more difficult than the introduction to proof but cited different reasons for their judgments. S4 indicated that the absence of sufficient example problems in his real analysis contributed significantly to its difficulty, where S2 found the concepts as well as proof construction more challenging in real analysis. Beyond these top-level judgments about "preparation," we found the two pairs of the participants (S1 & S3 and S2 & S4) provided two quite different narratives about the challenges of the course and how they had worked to address the challenges.

S1 and S3: Work Together, and Exploit Similarity Across Tasks

In explaining their success in real analysis, S1 and S3 both emphasized the quality of their instructor's teaching, citing four main similarities to instruction in their introduction to proof course: (a) group work in class, (b) regularly assigned and graded homework, (c) weekly quizzes, and (d) instructor encouragement. But this shared experience with instruction was coupled with changes in their learning activity. Where both S1 and S3 attended the Math Learning Center (MLC) at the university for the first time during their introduction to proof course and benefited from the activities and relationships supported there, neither attended in the MLC during real analysis. Instead, they worked remotely outside of class with the other members of their classroom small group that they maintained for the entire semester. When they got stuck on homework problems, they messaged with each other, sent pictures of the status of their solution attempts, and asked each other for suggestions. Both also reported they could reasonably predict the general nature of exam questions. They completed their homework each week, whose content predicted the weekly quizzes, which in turn predicted the content of exams. Their instructor also gave a practice final exam, described to resemble the actual final. S1 came to see a common structure among real analysis proofs (i.e., a standard way to develop and write epsilon-delta and epsilon-N arguments), whereas she could find no commonality among the proofs in her proof-based geometry course. When asked, S1 agreed that this common structure bore some similarity to her prior mathematics work, before the introduction to proof, of identifying known procedures and executing them on familiar tasks. S3 did not explicitly indicate an awareness of structural similarities among real analysis proofs but did strongly endorse the importance of collaboration with her peers, as complement to her own problem solving work on homework.

S2 and S4: Work Independently on Lots of Examples

In contrast, S2 and S4 emphasized the importance of repeated practice on numerous example problems for each course topic, as practice increased the likelihood of mastery and success on course assessments. In addition, both carried out this practice-focused work on their own. S4 expressed frustration that his instructor (different from S1-S3's) did not provide a sufficient number of examples comparable to his experience in his introduction to proof course. So he actively searched the internet for them, explaining that he looked for problems that were related to those worked in class and had complete solutions (proofs). S4 would then work the problem and compare his proof to the one provided. If he was unsure how to start, he reviewed the provided proof and then attempted to complete it on his own—comparing his proof to the one provided when he finished. S4 never went to the MLC during real analysis (in part because he did not think that Center personnel were prepared to help with that content), though he had done so regularly during his introduction to proof course. Also, he stated that he did not need to get help from MLC or office hours to complete the homework problems in real analysis, which are pretty much similar to what his instructor showed in class, whereas he was not able to even start some of the homework problems in the introduction to proof so he had to go to the MLC. S2 did not complain about the supply of example problems; she found the combination of problems worked in class, homework problems, and problems in the text not assigned for homework sufficient. Though she was part of the in-class group that S1 and S3 cited as important, S2 seldom contacted her group and solved most course problems on her own. She described her method of study for exams to involve "just doing lots of problems." Like her peers, S2 did not attend the MLC during real analysis, though she had done so repeatedly and productively during her introduction to proof course. She was also able to complete almost all the homework problems just from what her instructor showed her in class, whereas she reported that there were significant gaps between worked problems in class and homework, and between homework and exam problems in the introduction to proof.

Common Structure Among Real Analysis Proofs

One common thread in these results is the importance of noticing and abstracting a structure common to many real analysis proofs (what Selden & Selden [2013] have called a "proof framework"). Where S1 and S2 took different approaches to their work in real analysis, principally in how they engaged their peers, both spoke to the common structure they saw among the proofs their instructor produced and they produced in real analysis. S4 spoke to this issue in different terms, and S3 did so only obliquely and without emphasis. S1 saw the common structure among epsilon methods with some variation (e.g., epsilon-delta, epsilon-N) depending on the concepts involved in the statements (e.g., functions, sequence, and series). She was taught to always start with specific sentences in the structured way of proving the statements using the epsilon methods. She liked her instructor's practice of assigning similar problems using same approach/structure in homework and claimed her instructor's proof writing in class emphasized this pattern. Her perception of common structure contributed substantially her confidence going into major course assessments. S2 stated that the real analysis proofs were a lot more structured than the those in her introduction to proof course. She asserted this pattern ("the proof was basically the same for every type of like, every type of problem") and indicated that real analysis proofs had an "introduction" that stated an arbitrary epsilon, the body of the proof, and a "conclusion" that related the particular case to the definition. In contrast, S4 described the process of completing a real analysis proof after setting up its structure as "computation." He used that term to indicate the repeated process of determining appropriate values for delta or N in epsilon arguments. In his view, real analysis was 50% proof writing and 50% computation, where his abstract algebra experience was 80% of proof writing. Though he used different terms than S1 and S2, we interpret his assertion as similar to theirs: All three are citing structural regularities across many different real analysis proofs. This abstraction of common structure across many different proofs is significant for many reasons, not the least of which is that it narrows considerably the "problem solving space" students found themselves in during course assessments. None of the participants spoke to specific challenges in "filling in the blanks" of the common structure proof—Selden and Selden's (2013) "problem-centered part" of proof writing.

Discussion

This study produced three main results; all concern "outcomes" from one introduction to proof course. First, the introduction to proof course prepared all four participants relatively well for proof-based work in real analysis, one major content area of advanced mathematics. If the goal of such courses is to increase students' achievement in upper-level coursework, this course succeeded, at least for some students. Note that the introduction to proof course covered the basics of proof and proving and situated students' work in three different content areas. As such it fell into David and Zazkis's (2017) category of "Standard + Sampler" introduction to proof courses. Only five of the 176 courses they reviewed across the R1/R2 institutions in the U.S. were of this type. Second, even in our small sample, we have examples of students pursuing and achieving success in real analysis in different ways, even after "the same" introduction to proof. In particular, these four students took up group-work from their introduction to proof course in quite different ways—from substantially to not at all. Third, returning to our opening metaphor, mathematical work on the other side of the "proof door" can be similar in important ways to the computationally focus of their prior work. Three of the four participants reported regularities across real analysis proofs that resemble in some ways the mathematical work that preceded the focus on proof—recognize problems and apply the appropriate procedure to produce answers without significant feature of problem-solving. Though their introduction to proof course regularly asked these students to solve real problems, the tasks in real analysis significantly reduced the problematic nature of their mathematical work, as noted by S1, S2, and S4.

One major limitation of this study is our small and "correlated" sample. Three of the four participants experienced real analysis with the same instructor and engaged each other in the same small group—though they indicated no knowledge of their joint participation in the study. Our two different approaches to mastery (engage one's peers vs. repeated individual practice) are likely not the only narratives of mastering real analysis. Variation among students (e.g., in prior mastery experiences) and among instructors both likely contribute to the diversity of students' experience in real analysis. A second limitation leads to our next steps in this research: Most of the "graduates" of the introduction to proof course in this study have thus far had only modest experience in proof-based courses. Their journey will continue into new content areas and under the direction of different instructors. In the next phases of the research, we intend to track their experience in these new contexts (e.g., abstract algebra, real analysis II) and extend the reflective comparison of present and past experiences that we initiated in this study. We also hope to increase our sample size as more participants in our previous study enroll in proof-based content courses.

Reference

Benkhalti, A., Selden, A., & Selden, J. (2017). The saga of Alice continues: Her progress with proof frameworks evaporates when she encounters unfamiliar concepts, but eventually returns. In *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education*. San Diego, CA.

David, E., & Zazkis, D. (2017). Characterizing the nature of introduction to proof courses: A survey of R1 and R2 institutions across the U.S. In *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education*. San Diego, CA.

Selden, A. (2012). Transitions and proof and proving at tertiary level. In G. Hanna & M. de Villiers (Eds.), *Proof and proving in mathematics education,* New ICMI Study Series 15 (pp. 391–420). Springer.

Selden, A., & Selden, J. (2013). Proof and problem solving at university level. *The Mathematics Enthusiast*, *10*(1), 303–334.

Sellers, M., Roh, K. H., David, E., & D'Amours, K. (2017). A comparison of Calculus, Transition-to-Proof, and Advanced Calculus student quantifications in complex mathematical statements. In *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education*. San Diego, CA.

Smith, J. P., Levin, M., Bae, Y., Satyam, V. R., & Voogt, K. (2017). Exploring undergraduates' experience of the transition to proof. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 298–310). San Diego, CA.

Smith, J. P., & Star, J. R. (2007). Expanding the notion of impact of K–12 Standards-based mathematics and reform calculus programs. *Journal for Research in Mathematics Education, 38*, 3–34.