Reasoning about Quantities or Conventions: Investigating Shifts in In-service Teachers' Meanings after an On-line Graduate Course

Teo Paoletti
Montclair State University

Jason Silverman
Drexel University

Kevin C. Moore
University of Georgia

Madhavi Vishnubhotla
Montclair State University

Zareen G. Rahman
Montclair State University

Ceire Monahan
Montclair State University

Erell Germia
Montclair State University

Although pervasive in school mathematics, few researchers have paid explicit attention to the impact graphing conventions have on teachers' meanings for function and rate of change. We examine the role conventions play in in-service teachers' (ISTs') meanings and ways to promote their developing more sophisticated meanings. We provided pre and post surveys to ISTs enrolled in an on-line graduate course specifically designed to promote their development of more sophisticated meanings for function and rate of change via reasoning quantitatively. We prompted them to consider hypothetical student responses about these ideas in unconventional representations. In this report, we characterize ISTs' meanings in relation to conventions commonly maintained in school mathematics and examine shifts in the ISTs' meanings.

Keywords: Function; Rate of change; On-line education; In-service teachers

Whereas certain conventions (i.e., order of operations) impact the underlying mathematics at hand, other conventions are strictly representational choices (i.e., the input of a function is represented on the horizontal axis of a Cartesian coordinate system). Both types of conventions play an important role in mathematics but in this report we focus on the latter type of convention; although such conventions are pervasive in school mathematics (e.g., Hewitt, 1999), few researchers have examined the consequences for individuals' understandings of various ideas when particular conventions are strictly maintained. We are particularly interested in the extent to which teachers understand conventions as representational choices versus understanding these "conventions" as necessary features of particular mathematical ideas.

Thompson (1992) differentiated between a person using a "convention" unthinkingly and therefore being unaware of the "convention" as a convention versus understanding a convention as a particular choice that is customary (and often useful) while being aware that other choices may be equally correct or appropriate. Researchers have posited that students and teachers are hindered in making the latter distinction when they only have experiences in which particular conventions are maintained (e.g., Mamolo & Zazkis, 2012; Zazkis, 2008). Other researchers have noted that providing students opportunities to reason about relationships between quantities in non-canonical situations has the potential to support students in developing more sophisticated understandings that rely less on representational conventions and more on core mathematical ideas and understandings (e.g., Moore, Silverman, Paoletti, & LaForest, 2014).

In this report, we examine in-service teachers' (ISTs') function and rate of change understandings in relation to graphing conventions before and after an on-line course that was designed to support them in developing more sophisticated understandings of these ideas via reasoning about relationships between quantities (Thompson & Carlson, 2017). We address the questions: (a) To what extent do ISTs understand certain graphing conventions as choices or as mathematical rules that must be strictly maintained? (b) What impact does taking a graduate course focused on quantitative reasoning have on ISTs' meanings (and use of conventions)? As the intervention was on-line, we also seek to provide an existence proof that the impacts documented can be supported through carefully designed on-line professional development.

Theoretical Perspective

The on-line course in which this study is situated was designed to leverage ISTs' quantitative and covariational reasoning to support their developing more sophisticated mathematical meanings. Quantitative reasoning consists of an individual conceiving of a situation, constructing quantities as measurable attributes of objects, and reasoning about relationships between quantities (Smith III & Thompson, 2008; Thompson, 2011, 2013). When an individual conceives and coordinates two quantities together, they engage in covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Saldanha & Thompson, 1998). An increasing number of researchers have highlighted how students can leverage quantitative and covariational reasoning to develop understandings of various topical areas including function classes, rate of change, and the fundamental theorem of calculus (e.g., Confrey & Smith, 1995; Ellis, Ozgur, Kulow, Williams, & Amidon, 2015; Johnson, 2012; Thompson, 1994a, 1994b) and to enact important mental processes such as generalizing and modeling (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson, Larsen, & Lesh, 2003; Ellis, 2007).

Of relevance to this report, Moore et al. (2014) highlighted the extent to which engaging students in reasoning about relationships between quantities can support students in developing mathematical understandings that are not constrained by conventions commonly maintained in school mathematics (i.e., representing the input of a graphically represented function on the horizontal axis with the variable x). The researchers outlined several principles teacher educators can use to support pre-service teachers (PSTs) and ISTs developing more sophisticated meanings including (a) using tasks that intentionally break from conventional representational systems, (b) routinely using quantitatively rich situations (i.e., situations in which an individual can construct and reason about a variety of quantities in order to solve a problem), and (c) maintaining an explicit focus on quantities and their relationships in classroom discourse.

Relevant Literature

Students' and Teachers' Convention Understandings

Several researchers have noted that students and teachers can develop insufficient mathematical understandings if certain conventions are strictly maintained in school mathematics (Mamolo & Zazkis, 2012; Thompson 1992; Zazkis, 2008). For example, researchers who have investigated students' meanings for function and rate of change (e.g., Akkoc & Tall, 2005; Montiel, Vidakovic, & Kabael, 2008; Moore et al., 2014; Oehrtman, Carlson, & Thompson, 2008) have found that students often maintain meanings that require certain representational conventions to be followed. With respect to students' function meanings, Montiel, Vidakovic, and Kabael (2008) identified students applying the vertical line test, a common procedure included in U.S. curricula, to determine if a graph defined by $r = 4$ represented in the polar coordinate system represented a function. Breidenbach, Dubinsky, Hawks, and Nichols (1992) illustrated that only 11 of 59 undergraduate students in their study understood a graph we interpret as representing the function $x = f(y) = \sin(y)$ for $-4 < y < 4$ with x and y represented on the horizontal and vertical axis respectively as representing a function (i.e., *x* as a function of *y*). In these examples, the researchers posed graphs they intended to represent functions but the students' meanings did not afford such interpretations; one possible explanation for this observation is that the students understood representational choices (e.g., graphs are unquestionably represented in the Cartesian coordinate system with the independent quantity represented by *x* on the horizontal axis) as mathematical rules that must be followed.

Moore et al. (2013, in preparation) highlighted the extent to which PSTs in their study understood function and rate of change in relation to graphing conventions. The researchers noted less than 36% of PSTs interpreted hypothetical student work as unquestionably correct when these responses used unconventional, but mathematically viable graphs. Many PSTs indicated the hypothetical student would be correct if a certain feature of the graph was changed to maintain conventions but concluded that in the given orientation the hypothetical student was incorrect. We extend Moore and colleagues (2013, 2014, in preparation) work by examining a different population's, ISTs', function and rate of change understandings in relation to graphing conventions. We also examine the extent to which an on-line course focusing on reasoning quantitatively has the potential to promote shifts in ISTs' meanings.

Teaching and learning mathematics on-line. Online courses at the university level continue to grow as there is a belief that such courses can reduce expenditure and increase enrollment (Allen, Seaman, Poulin, & Straut, 2016). In this study, we employed an instructional environment grounded in design-based research that is referred to as Online Asynchronous Collaboration (OAC) in Mathematics Teacher Education (Silverman & Clay, 2010). At its core, the OAC model is grounded in the belief that replicating traditional teaching practices is not sufficient for online learning environments (Reeves, Herrington, & Oliver, 2004). The implementation of the OAC we report here consists of iterative cycles of three to four day "private" problem solving in an on-line discussion board (viewable only by the individual student and instructor), then three to four days of "public" discussion in which all students are given the opportunities to read, comment on and ask questions about each other's solutions. The last few days of each unit are designed to support students' synthesis and reflection on the ways of reasoning each problem set was designed to highlight. Researchers (Silverman, 2011; Silverman & Clay, 2010) have shown that this OAC model has the potential to support ISTs' development of pedagogical content knowledge and mathematics knowledge for teaching; we extend these results by examining how this model has the potential to support teachers' developing more sophisticated understandings in relation to graphing conventions.

Methods and Analysis

Participants and Settings

The ISTs who participated in the study were enrolled in a fully online graduate mathematics program designed specifically for ISTs. The ISTs were geographically distributed across the U.S. and each was, at the time of the study, a 6-12 grade mathematics teacher who was certified to teach mathematics in his/her home state. All of the ISTs had completed a minimum of three mathematics courses beyond Calculus III and had an undergraduate GPA of 3.0 or better. In total 34 ISTs took both the pre and post survey.

The on-line course was designed with the intention of leveraging the teachers' quantitative and covariational reasoning to develop more sophisticated understandings and followed the recommendations put forth by Moore et al. (2014) outlined above. The initial unit asked the ISTs to track and describe the behavior of various contextualized relationships (i.e., a car driving back and forth along a road as described by Saldanha and Thompson (1998)). There was a particular focus on supporting ISTs in identifying quantities from a given context, using variables to represent varying quantities, and analyzing relationships between relevant quantities verbally, numerically, and graphically. The remainder of the term asked ISTs to leverage these skills with a focus on exploring a variety of functional relationships (e.g., polynomial functions, trigonometry, related rates problems, modeling, and ideas from calculus) from a quantitative perspective. Table 1 presents an overview of the 10-week course.

Table 1. 10-week Course Overview

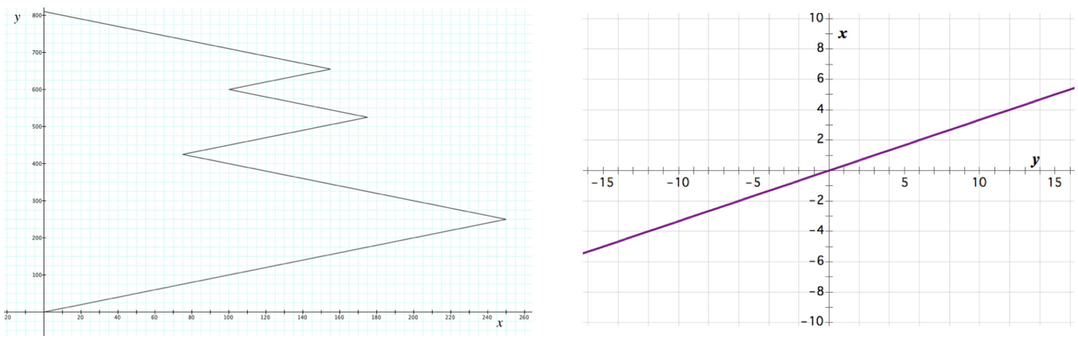
Week	Focus
1	Covariation of Quantities
2	Trigonometry
3	Periodicity and Covariation: Trigonometric Functions
4	Functions as Relationships in Context
5	More Functions as Relationships/Functions as Actions and Processes
6	Families of Functions
7	Average Rate of Change
8	Rate of Change and Rate of Change Functions
9/10	Covariation in the Classroom

Analysis

We coded the ISTs' responses using open and axial approaches (Strauss & Corbin, 1998) and thematic analysis (Braun & Clarke, 2006). Throughout the coding process, the researchers did not know which IST's response they were coding or if the IST's response was part of the pre or post survey. A member of the research team read a subset of IST responses and we met to discuss our observations, identify commonalities across responses, and adapt or create new codes to capture more responses. We iterated this process four times as we refined our codes to accurately capture all responses; as the resulting codes are both methods and results, we present the codes themselves in the results. After we agreed on a final set of codes, a second researcher recoded approximately 65% of the data. We calculated inter-rater reliability by comparing the number of times both coders agreed on a code, achieving a high level of agreement on each problem (Sideways Mountain Task, Kappa = 0.78 and $y = 3x$ Task, Kappa= 0.85).

Task design. We adapted tasks used by Moore et al. (2013, submitted) to make inferences about PSTs' understanding of function and rate of change in relation to graphing conventions into items ISTs responded to in pre and post-course on-line surveys. Each task was designed with the intention of examining ISTs' understanding of mathematical ideas in relation to graphing conventions. In order to ensure the ISTs noticed the unconventional nature of the graphs, the tasks included hypothetical student responses that deviate from a particular convention but are mathematically viable (from the researchers' perspective). For example, the Sideways Mountain Task prompts an IST to respond to a student who stated for the graph in Figure 1a that "Sure, it can be a function… x is a function of y." Whereas from the researchers' perspective the students' statement is mathematically correct, the graph, in its given orientation does not pass the vertical line test, which as described above, is often critical to students' and teachers' meanings for function in a graphing context. Hence, the tasks allow us to examine the extent to which an IST's function understandings are related to particular graphing conventions (i.e., a function's input must be represented by the variable x or on the horizontal axis, or both).

Like the Sideways Mountain Task, the $y = 3x$ Task supports our examining ISTs' rate of change understandings in relation to graphing conventions. The task prompts ISTs to consider a student who graphed the relationship $y = 3x$ as shown in Figure 1b. Although the graph does represent the relationship defined by $y = 3x$, the hypothetical student's work deviates from the convention of representing x and y on the horizontal and vertical axes, respectively. Hence, the task provides insights into the extent to which ISTs' meanings for graphs and rate of change rely on representing particular variable quantities on particular axes versus accurately representing relationships between two quantities.



(a) (b)

Figure 1. (a) Sideways Mountain Task: Is x a function of y? (b) The $y = 3x$ Task: A hypothetical student's work.

Results

In this section we first describe the codes we created to capture the ISTs' responses. We then compare the ISTs' pre and post survey results for each of the two tasks described above. For both tasks, our final coding scheme categorized the extent to which the ISTs interpreted the hypothetical students' mathematical statement as viable. This analysis provides insights into the extent to which the ISTs' meanings for graphs, function, and rate of change are rooted in reasoning about relationships between quantities versus requiring graphing conventions to be maintained. Demonstrating a focus on understanding statements concerning rate of change and function to be statements about relationships between quantities, the first code was for responses that indicated the student's mathematical statement is correct notwithstanding the student breaking from conventions. Indicating a tension between reasoning about relationships between quantities and graphing conventions, the second category was for responses that specified the student's statement was mathematically true but, despite this, the student's solution was wrong because he or she did not follow conventions. Signifying the ISTs' meanings required certain conventions to be maintained, the final category was for responses that either indicated the student's mathematical statement was incorrect or did not address the student's statement.

Table 2 presents the code description, an example response to the Sideways Mountain Task and the counts for the pre and post survey. We first highlight that prior to the course, a majority of the ISTs interpreted the hypothetical student's solution as incorrect, despite the student's statement being mathematically viable from our perspective. Second, we note the trend of a positive shift in ISTs' responses towards interpreting the student's mathematical statement as correct despite the student breaking from conventions after taking the on-line course. We take this to indicate that the course supported many of the ISTs in developing more sophisticated meanings in regards to functions and their graphs. Finally, we note that despite this trend, nine ISTs still interpreted the student's mathematical statement as *incorrect* or did not address the students' mathematical statement in the post-survey. We return to this observation in the implications.

Table 2. Code descriptions, sample responses, and counts for the pre and post survey for the Sideways Mountain Task.

Code description (value)	Example Responses to the Sideways Mountain Task	Pre	Post
The student's mathematical statement is correct despite breaking from conventions. (1)	That's great! I am so glad you were able to apply the "vertical line test" in a horizontal orientation and realize that you would have a function. You are correct in saying that x is a function of y.	11	19
The student's mathematical statement is true but the student is incorrect because he/she broke from conventions. (2)	I think the student is understanding that x can be a function of y but they are not displaying it correctly through the graph.	5	6
The student's mathematical statement is incorrect or the IST did not address the student's mathematical statement. (3)	It was not a good explanation and x is not a function of y, y is a function of x. The value of y depends on x. They also did not describe what would make it a function.	18	9

Table 3 presents the code description, an example response to the $y = 3x$ Task and the counts for the pre and post survey. We again highlight that there is a general trend towards more ISTs' responses indicating that the hypothetical student's response is correct despite breaking from conventions. In contrast to the responses to the Sideways Mountain Task, we note that a majority of ISTs' pre-survey responses indicated that the student's statement was correct before the intervention. We take this finding to indicate that the ISTs' meanings for rate of change may be less reliant on certain conventions being maintained prior to taking the on-line course as compared to their function meanings.

Table 3. Code descriptions, sample responses, and counts for the pre and post survey to the $y = 3x$ Task.

Code description (value)	Example Responses to the $y = 3x$ Task	Pre	Post
The student's mathematical statement is correct despite breaking from conventions. (1)	In this case, the student has graphed the relationship correctly given their choice of axis. Technically there is absolutely nothing wrong with this graph.	20	24
The student's mathematical statement is true but the student is incorrect because he/she broke from conventions. (2)	The student cannot receive full credit, as the graph is wrong, however it can easily be fixed by discussing the y as the vertical axis and the x as the horizontal axis. Once this discussion has ensued, I would ask the student to graph again but prompt them that they were correct in their understanding of y being equal to 3 times the given x value.	6	8
The student's mathematical statement is incorrect or the IST did not address the student's mathematical statement. (3)	The student did not graph the slope correctly, instead of a positive 3 they graphed a negative 3. They did label their x and y-axis. Therefore, they are showing some correlation as to how the values of x and y vary and covary with each other.	8	2

Comparing surveys. To further examine shifts in the ISTs' meanings from the pre to post survey, we assigned numerical values to each of the categories (shown in parentheses in the code description column). Table 4 presents the pre and post averages for each task; a score closer to 1 indicates that on average, the ISTs were attending more to the underlying quantitative relationships than to the student's response adhering to graphing conventions. In order to examine if there were statistically significant differences between the ISTs' responses pre and post course, we conducted one-tailed Wilcoxon Signed-Rank tests to examine if the mean scores differed significantly. We conducted one-tailed test because we expected the ISTs would exhibit a positive shift in their meanings based on the intervention and we conducted Wilcoxon Signed-Rank tests rather than *t*-tests as we cannot say if the population is normally distributed. Table 4 presents the *p*-values for each test. We note that there was a statistically significant result for the Sideways Mountain Task but not for the $y = 3x$ Task. We conjecture the latter observation may be due to the fact that the ISTs' initial responses indicate a tendency to evaluate the student's statement in the $y = 3x$ Task as correct prior to the on-line course.

Table 4. Average scores of pre and post survey for ISTs and *p*-values from a Wilcoxon Signed-Rank test.

	Sideways Mountain Task	$y = 3x$ Task
Pre	2.21	1.65
Post	1.71	1.35
p-value	0.0037*	0.0618

Discussion and Implications

In this report, we make several contributions to the research examining ISTs' understandings of mathematical ideas and ways to support ISTs' quantitative reasoning. We demonstrated many ISTs' initial meanings for function required certain graphing conventions to be maintained which is largely compatible with the PSTs reported by Moore and colleagues (2013, submitted). This finding underscores the importance of addressing such meanings in professional development and in PST training programs as teaching experience is not enough to support teachers in developing these meanings. We also highlight, and compatible with the PSTs reported by Moore and colleagues, the ISTs' responses to the $y = 3x$ Task differed from their responses to the Sideways Mountain Task. This finding highlights the extent to which an individual is constrained by a particular convention (i.e., a function's input is represented on the horizontal axis by the variable x) may be idiosyncratic to the particular mathematical idea at hand. Some may interpret this finding to indicate ISTs' and PSTs' meanings for rate of change are more focused on the underlying relationship between quantities (i.e., reason quantitatively) rather than maintaining particular conventions. Before we make such an argument, we believe there needs to be more research investigating teachers' understandings of rate of change in other non-canonical situations (i.e., polar coordinates).

Researchers (e.g., Mamolo & Zazkis, 2012; Moore et al., 2014; Paoletti, Stevens, & Moore, 2016; Thompson, 1992) have indicated that educators should provide students, PSTs, and ISTs with repeated opportunities to address unconventional situations in order to support them in expanding their meanings for various mathematical ideas such that they understand what aspects are conventional and what are required mathematically. Our data provides an existence proof that an on-line course can provide such opportunities for ISTs. This finding is especially important as on-line interventions have the potential to be scalable in ways that face-to-face courses typically are not. Future researchers may be interested in implementing and studying such scaling efforts to improve teachers' mathematical meanings.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. DRL-1350342 and DRL-1222355. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the NSF.

References

Akkoc, H., & Tall, D. (2005). *A mismatch between curriculum design and student learning: the case of the function concept*. Paper presented at the Proceedings of the sixth British Congress of Mathematics Education, University of Warwick.

Allen, I., Seaman, J., Poulin, R., & Straut, T. (2016). *Online report card: Tracking online education in the United States*. Babson Park, MA: Babson Survey Research Group.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*(3), 247-285.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Ginsburg, H. P. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. New York, NY: Cambridge University Press.

Hewitt, D. (1999). Aribtrary and Necessary Part 1: A way of viewing the mathematics curriculum. *For the Learning of Mathematics, 19*(3), 2-9.

Mamolo, A., & Zazkis, R. (2012). Stuck on convention: A story of derivative-relationship. *Educational Studies in Mathematics, 81*(2), 161-167.

Meira, L. (1995). The microevolution of mathematical representations in children's activity. *Cognition and Instruction, 13*(2), 269-313.

Montiel, M., Vidakovic, D., & Kabael, T. (2008). Relationship between students' understanding of functions in cartesian and polar coordinate systems. *Investigations in Mathematics Learning, 1*(2), 52-70.

Moore, K. C., Liss, D., Silverman, J., Paoletti, T., LaForest, K. R., & Musgrave, S. (2013). Pre-service teachers' meanings and non-canonical graphs. In M. Martinez & A. Castro Superfine (Eds.), *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 441-448). Chicago, IL: University of Illinois at Chicago.

Moore, K. C., Silverman, J., Paoletti, T., & LaForest, K. R. (2014). Breaking conventions to support quantitative reasoning. *Mathematics Teacher Educator, 2*(2), 141-157.

Oehrtman, M., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. L. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27-42). Washington, D.C.: Mathematical Association of America.

Paoletti, T., Stevens, I., & Moore, K. C. (2016) “We learned a trick...” *Mathematics Teacher*, 110(6), 446-453.

Reeves, T., Herrington, J., & Oliver, R. (2004). A development research agenda for online collaborative learning. *Educational Technology Research and Development, 52*(4), 53-65.

Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen, K. R. Dawkings, M. Blanton, W. N. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the 20th Annual Meeting of the North American Chapter of the International Group for the*

Psychology of Mathematics Education (Vol. 1, pp. 298-303). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Silverman, J. (2011). Supporting the development of mathematical knowledge for teaching through online asynchronous collaboration. *Journal of Computers in Mathematics and Science Teaching, 30*(1), 61-78.

Silverman, J., & Clay, E. L. (2010). Online Asynchronous Collaboration in Mathematics Teacher Education and the Development of Mathematical Knowledge for Teaching *The Teacher Educator, 45*(1).

Smith III, J. P., & Thompson, P. W. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the Early Grades* (pp. 95-132). New York, NY: Lawrence Erlbaum Associates.

Strauss, A. L., & Corbin, J. M. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd ed.). Thousand Oaks: Sage Publications.

Thompson, P. W. (1992). Notations, conventions, and constraints: Contributions to effective uses of concrete materials in elementary mathematics. *Journal for Research in Mathematics Education, 23*(2), 123-147.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e* (pp. 33-57). Laramie, WY: University of Wyoming.

Thompson, P. W. (2013). In the absence of meaning. In K. Leatham (Ed.), *Vital directions for research in mathematics education* (pp. 57-93). New York, NY: Springer.

Thompson, P. W., & Carlson, M. P. (2017). Variation, Covariation and Functions: Foundational Ways of Thinking Mathematically. In J. Cai (Ed.), *Handbook of Research in Mathematics Education*. Reston, VA: National Council of Teachers of Mathematics.

Zazkis, R. (2008). Examples as tools in mathematics teacher education. In D. Tirosh & T. Wood (Eds.), *International handbook in mathematics teacher education, vol. 2: Tools in mathematics teacher education* (pp. 135-156). Rotterdam, Netherlands: Sense.