Conceptual Blending: The Case of the Sierpinski Triangle Area and Perimeter

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In this report, we present an analysis of 10 individual interviews with graduate mathematics education students about the area and perimeter of the Sierpinski triangle (ST). We use conceptual blending as a theoretical and methodological tool for analyzing students' reasoning to investigate how students encounter and cope with the ST having zero area and infinite perimeter. Our analysis documents the diverse ways in which the students reasoned about the situation. Results suggest that conceptualizing an infinite perimeter is more accessible to these students than is zero area, that encountering the paradox is dependent on how blends are composed, and that resolution of the paradox comes through completion and elaboration. The analysis furthers the theoretical/methodological framing of conceptual blending as a useful tool for revealing the structure and process of student reasoning.

Keywords: Conceptual blending, Infinite processes, Fractal, Paradox, Student thinking

It's still hard for me to wrap my mind around the Sierpinski triangle, and that there's infinite perimeter and no area. It makes sense to me individually, but both together at once, I'm still, it's still mind-boggling. – Carmen, graduate mathematics education student

Straightforward notions of the area and perimeter of geometric shapes are first learned in elementary school, and are revisited and leveraged throughout middle and high school. When dealing with fractals, however, some counter-intuitive situations involving these ideas arise. One such situation, a region with zero area and an infinitely long perimeter, was encountered by a class of mathematics education master's degree students in a chaos and fractals course when investigating the Sierpinski Triangle (ST) shown in Figure 1. As seen in Carmen's introductory quote, this was a non-trivial exercise and caused some students serious consternation.

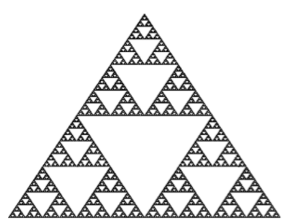

Figure 1. The sixth step in creating the Sierpinski Triangle.

To investigate student reasoning about the ST we conducted individual interviews about three weeks after its in-class investigation. Based on the interview data, and using the ideas of conceptual blending, we address the following two related research questions: (1) How do students make sense of (a) area and (b) perimeter of the ST? (2) How do students coordinate the area and perimeter of the ST and cope with the resulting paradoxical situation?

Theoretical Background

We use conceptual blending theory (Fauconnier & Turner, 2002; Núñez, 2005) as a theoretical and methodological tool for analyzing students' coordination of two infinite processes, one increasing (perimeter) and one decreasing (area). Blending is based on the notion of mental spaces, which are "small conceptual packets constructed as we think and talk, for the purposes of local understanding and action" (p. 40). According to the theory, these mental spaces "organize the processes that take place behind the scenes as we think and talk" (p. 51). Conceptual blending is defined as the conceptual integration of two or more mental spaces to produce a new, *blended,* mental space. An important feature of this new blended space is that it develops an emergent structure that is not explicit in either of the input mental spaces. This emergent structure is generated by three processes: composition, completion, and elaboration.

Composition is the selective projection of elements from input spaces into a common space. During composition, distinct elements may be projected on top of each other or *fused,* and common elements may be projected separately. The composition process develops a new space, with the potential for structure not available in either input space. *Completion* is the process of recruiting familiar frames to the blended space, along with their entailments. That is, an individual recognizes certain aspects of a blended space as parts of a familiar frame and brings in additional knowledge, scripts, assumptions, etc., to complete the frame and prescribe structure for the blended space. These frames can serve as tools for *elaboration*, which is sometimes called *running the blend*. Elaboration is the process that leads to the emergence of something new within the blended space, using the tools of the completion process and the elements that compose the blend. These processes, composition, completion, and elaboration, do not necessarily take place sequentially (Fauconnier & Turner, 2002).

Underlying our analyses is our knowledge of previous research related to conceptual blending in other contexts (e.g., Lakoff & Núñez, 2000; Yoon, Thomas, & Dreyfus, 2011; Zandieh, Roh, Knapp, 2011), infinity (e.g., Ely, 2011; Fischbein, Tirosh, & Hess, 1979; Núñez, 2005), paradox (e.g., Dubinsky, Weller, McDonald, & Brown, 2005ab; Sacristán, 2001; Wijeratne & Zazkis, 2015). A review of these works is beyond the scope of this report, but we acknowledge the impact of this prior work for our own and note that our work is some of the first to bring together all these ideas.

Methods

The study took place in a graduate level mathematics course of 11 mathematics education students (10 of whom agreed to participate in individual interviews). The course was taught by one of the research team members. Students sat in four groups and daily worked on tasks in their small groups and engaged in whole-class discussions of these same tasks. Data was collected as part of a larger study and included video-recordings of each class session, individual task-based interviews conducted at the middle and end of the semester, and copies of all student work.

The focus of the analysis in this paper are students' responses to the following question from the mid-semester interview: *In class, we discussed the Sierpinski Triangle. How do you think about what happens to the perimeter and the area of the ST as the number of iterations tends to infinity?* This question was accompanied by a printout of the ST (as seen in Figure 1), with a follow-up prompt to tell us what they thought about the following claim of a fictitious student, Fred: "The computation shows that the perimeter goes to infinity because the perimeter is given by $3(3/2)^n$ which increases to infinity as n tends to infinity. But, the perimeter can't really be infinitely long, because there is nothing left to draw a perimeter around, since the area goes to zero." This interview task was designed based on the classroom discussion of the ST, which took place two weeks before we began interviewing students. At that time, students seemed to agree that the area went to zero but were unsure of what happened to the perimeter. They publicly considered the possibilities that it went to infinity, converged to some value, or did not exist because there was nothing left for a perimeter to go around. The interview was structured so that we would first gain insight into the students' reasoning about the area and perimeter of the ST, followed by an opportunity for them to respond to Fred's claim.

To identify a student's input space for area (similarly for perimeter), we first marked which of their utterances were about the area. Next, we categorized these utterances into sets of ideas about the area of the ST - including the process by which it is created and the resulting product. In the spirit of grounded theory (Strauss & Corbin, 1998), these ideas were coded and compared iteratively until a coherent set of idea codes emerged. The interviews were divided into two groups and analyzed by different members of the research team. These analyses were then swapped, compared, and vetted.

We investigated students' blending by identifying each of the three processes: composition, elaboration, and completion. To see how a student's blend was composed, we identified which elements of the student's input spaces were brought up as they considered the coordination of area and perimeter (prompted by Fred's paradox). We identified the ways students elaborated their blended spaces by identifying ideas which were not in the input spaces, but emerged as they worked to make sense of the task. Interpretation of completion and elaboration was done first as a group, with all four authors debating each point, then a more detailed pass was made by two members of the team in close comparison with the transcripts, and these analyses were then discussed again among the four authors until agreement was reached.

Sample Results

During the in-class discussions about the ST there was widespread agreement that the area would go to zero but less agreement that the perimeter would diverge to infinity. We were therefore surprised to find that only six of the ten students concluded that the area of the ST goes to zero but all ten students concluded that the perimeter tends to infinity.

Area and Perimeter (Research Question 1)

Among students' justifications for their conclusions, we identified seven qualitatively different mental space elements for area and seven qualitatively different mental space elements for perimeter. As a sample we display three of the most prominent different mental space elements side-by-side, with descriptions of the elements and illustrative quotes.

Area	**Perimeter**
<u>Infinite decreasing process</u> Common among all 10 students was the element that area is the result of an infinite, decreasing process. For example: Carmen: *So, ok eventually the area gets to zero, but that's if you could do it infinitely many times. And if you actually*	<u>Infinite increasing process</u> All 10 students conceived of the perimeter of the ST as the result of an infinite increasing process. Elise's reasoning is typical of this thinking: Elise: *You're just like forever adding length to your perimeter, so I feel like your perimeter is forever increasing.*

conceptualize doing infinitely many times you're never gonna stop.	
<u>Area removed at each step</u> All students except Curtis there was explicit use of the justification that area is removed at each step. Two students computed the first few steps during their interviews. Kay: *We're always taking out the middle triangle of each equilateral triangles and we're doing that infinitely so it's like we're taking away area with each iteration.*	<u>Perimeter is added at each step</u> All students except Curtis also pointed to the fact that perimeter is added at each step. Four students accompanied this with computation for the first few iterations. Joy: *I think it goes towards infinity because each iteration you're creating more triangles and so you're creating, you're adding to the perimeter.*
<u>Change in the rate of change</u> Shani and Kay, who were in the same group, were the only two students who concluded that the area tended to something non-zero. They were also the only two who shared what we refer to as the change in the rate of change for area element, as exemplified in this excerpt. Shani: *As we keep taking off little pieces and more become white, it's getting smaller and smaller. Or the amount that it's increasing is getting smaller and smaller and smaller.*	<u>Change in the rate of change</u> Two other students, Elise and Carmen, gave some consideration to the rate at which the perimeter increases and to changes in this rate. For example, Elise argued that Elise: *Every time after the first iteration I'm adding more perimeter than I added before. So if I keep adding more then I think it's going to keep going to infinity because I'm just going to keep adding bigger and bigger.*

Discussion. Other reasoning about area and perimeter included reasoning multiplicatively, reasoning about congruent figures, reasoning with geometric series and associated convergence or divergence criteria, and thinking of the ST being composed of leftover or removed pieces. Students primarily made sense of the area and perimeter of the ST as infinite iterative processes. This is not surprising given the construction process students were introduced to in class. What *did* surprise is the fact that, except for Curtis, students used informal additive reasoning to reach their conclusions. The few students who did some computations did so only for the first few iterations and did not generalize the adding of perimeter or removal of area into algebraic expressions from which to take limits. While some students used limit language or referred to convergence criteria, it was not done concretely, despite their mathematics experience.

Given students' informal ways of reasoning, the parallelism between area and perimeter ideas is noteworthy. Each element of reasoning about area had a corresponding element of reasoning about perimeter. While some of these ideas were common (infinite processes, adding area, removing perimeter), others were not. In several cases students' idiosyncratic ways of thinking were consistent within students across area and perimeter. Despite the idiosyncrasies, there was quite a lot of consistency in ways of reasoning across students, both with respect to area and with respect to perimeter.

Blending Area and Perimeter (Research Question 2)

One element appears in every student's blended space which did not appear in the area/perimeter section: *infinite creation process*. This element is a result of fusion, wherein two input space elements (here, infinite increasing and infinite decreasing) are projected onto one element. As students were introduced to the ST as something created through an iterative, recursive process affecting both area and perimeter, in a sense the students are re-fusing elements which they originally separated. To organize these ideas, a three-part diagram is used: rectangles represent mental spaces, with the upper rectangles representing the input mental spaces and the lower rectangle representing the blended mental space, and the lines show mappings between the spaces. Due to space constraints, we present only four students and two blending diagrams.

Joy. We gained access to Joy's blending process primarily through her response to Fred's argument. Her blended space was composed of the infinite process of creating the Sierpinski Triangle, the area tending to zero, and perimeter tending to infinity. Completion brought into the blended space a metaphor of *perimeter as fence*, along with several entailments. One such entailment is that fences should remain, even if the space they enclose is no longer there. Part of Joy's elaboration based on this frame, as she worked to resolve Fred's paradox, was to say that "we don't count their space, but there is still a perimeter associated with it." Another entailment of the fence framing is that not only do fences have length, but they also take up space. This contributed to another element of Joy's elaboration, that the perimeter will fill in the Sierpinski Triangle, "so eventually in a sense it's all fence." Some parts of Joy's elaboration are grounded in a physical metaphor, and she recognizes this when responding to Fred. She adds to her elaboration that the Sierpinski Triangle is "not a real object," and identifies the juxtaposition of an infinite mathematical process with the physical world as "where the disconnect is."

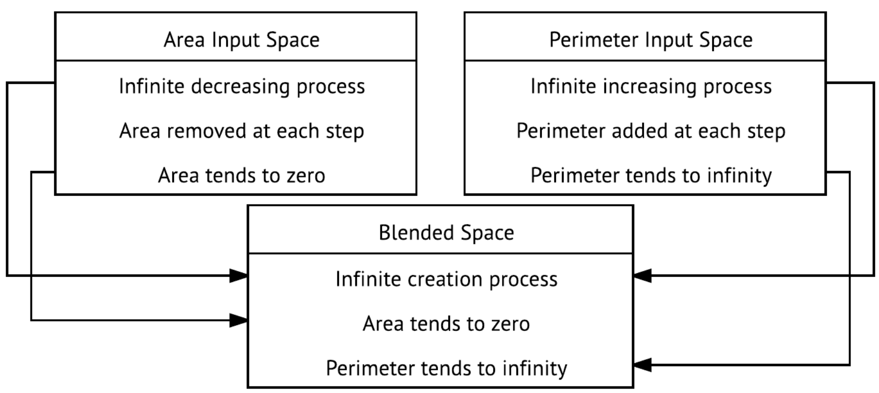

Figure 2. Blending diagram for Joy's reasoning

Elise. Like Joy, Elise's blended space is composed of the infinite process of creation for the ST, perimeter tending to infinity, and area tending to zero. However, the framing metaphor that completes Elise's space is one of a skeleton, not a fence. She elaborated her blend, saying, "I'm thinking of our perimeter as like, like I guess I think at the end of this I have this skeleton, so I have no area, nothing is left inside" This skeleton metaphor brings with it entailments of bones remaining when flesh has gone, clearly mapping perimeter to bones and area to flesh. In addition, we note that Elise mentions "at the end" in her elaboration, perhaps hinting that she sees the ST as an abstract object at the end of a generating process.

Curtis. As with Elise and Joy, Curtis's blended space is composed of an infinite creation process, perimeter tending to infinity, and area tending to zero. Unique to his blended space, however, is his formal, multiplicative formulation of area and perimeter as the limits of infinite sequences. The completion process brings in a zooming frame, saying, "we could say you could zoom in for infinitely, as much as you want, and you could get like these as tiny and tiny as you want, there's still more perimeter to draw" when prompted with Fred's paradox. The second frame we see Curtis leverage is one related to mathematics classes (e.g., Calculus, Analysis) where symbolic manipulations are sufficient. Evidence of this comes from the fact that Curtis did not encounter a paradox when considering an object with zero area and an infinite perimeter on his own, something he elaborated by saying "this isn't like, not physically drawing something like a perimeter, it's kind of just a concept."

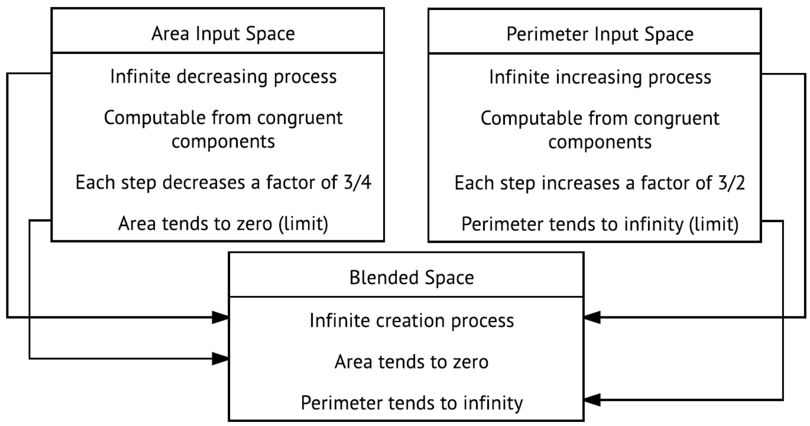

Figure 3. Blending diagram for Curtis's reasoning

Carmen. Carmen's blended space is, like several others', composed of an infinite process of creation, area tending to zero, and perimeter tending to infinity. The completion of her blend, however, is particularly distinct. She brings in a calculus frame and identifies "analogies to calculus or real analysis," including Riemann sums, that she sees as similar to Fred's paradox. The "calculus arguments" that she references seem to imply, to Carmen, that Fred's paradox is like other paradoxical situations that she has seen in previous mathematics courses. Upon reading Fred's arguments during the interview, Carmen stops to query whether "the perimeter can't really be infinitely long" implies zero perimeter or some non-zero finite length (for Fred). She proceeds to resolve the dilemma by eliminating each, leaving only the possibility that the perimeter is indeed infinite and Fred is wrong. During this episode, two more frames appear. Like Joy, she brings in a fence metaphor for the perimeter and the entailment that fencing should remain, but does not use the idea that fences take up space. Her elaboration using the fence frame, "you have sort of your old triangle fences that you had before [...] we still have this fence around, that big triangle and the center, and we still have those other ones we made before," is how she argues that the perimeter of the ST cannot be zero. Finally, she brings the frame of self-similarity, with the entailment that "we can keep zooming in." The elaboration using this frame is that the perimeter cannot be a finite value, which she explains using a contradiction. Carmen says "I think if we could [stop] then you could say ok it's this number," but the zooming goes on forever, "so that's kind of why it can't be a number."

Discussion. Our analysis of students' blending processes, especially as provoked by encountering Fred's argument, revealed how students deal with the paradox of coordinating infinite perimeter and zero area associated with the ST, and how they cope with, or resolve, the cognitive dissonance it provokes. It was sometimes challenging to unpack and distinguish the completion and elaboration processes. We attribute this difficulty in part to the fact that this was the second opportunity in which the students were prompted by Fred's paradox.

All students *composed* a blended space from their area and perimeter input spaces following Fred's prompt, and most of them also *completed* their blended space with additional frames, which then supported *elaboration* of the blend - leading to new implications. In two students' interviews we saw evidence of completion but not elaboration; only for one student we do not have evidence of completion. We saw one commonality across all students' composition processes: the fusion of *infinite (increasing) process* and *infinite (decreasing) process* into a unified *infinite creation process* for the stepwise creation of the ST. This is not to say that there was a shared conception of exactly what happens at each step, only that the process is infinite.

For all but one student, we have evidence of 1-3 distinct frames being used to complete their blended spaces. In all the cases, one of the frames has to do with the nature of mathematics – e.g., the nature of infinite processes. However, four students also used physical frames (fence, skeleton, zooming-in) and their entailments to coordinate area and perimeter and to make sense of that coordination.

Conclusion

As the 10 students we interviewed were in the same graduate program, part of the same class, had worked together and discussed the Sierpinski Triangle (and, essentially, Fred's argument), we expected to see a certain level of consistency in their responses. However, this was not entirely the case, as seen at every stage of our analysis. To be sure, some ideas about the nature of the infinite iterative process were present in all interviews. But while in class students seemed comfortable with the idea that the area of the ST goes to zero, and concerned about what happens to the perimeter, *all* students' input spaces for perimeter included that it was infinite, and only six of the ten spaces included area going to zero. There were other idiosyncratic elements present in students' input spaces such as Curtis's multiplicative reasoning. There were also idiosyncrasies in terms of the composition of blended spaces. Some students completed their blends with ideas from calculus or analysis, fractal dimension, and metaphors. These frames resulted in varied elaborations. Some related to the nature of the ST, such as "it's not a real object", its non-integer dimension, or that is only the remaining outline; others framed the nature of the paradox itself.

More generally, our analysis methods allow us to point to some of the precise points of departure, from initial ideas to completing frames and final elaborations, one of the methodological implications of our work for future researchers. Along with Zandieh et al. (2014), our articulation of the component process of conceptual blending in a mathematical context allow for nuanced analysis of students' reasoning – though they looked at group blends and types of blends, while we look at more individualistic reasoning. This is particularly relevant for situations where students must bring together multiple ideas. Identifying all three processes - composition, completion, and elaboration - allows us to examine not only the main ideas students mention, but how they are used and enacted, or what leverage they give students in thinking about mathematical objects. This is in contrast to other lenses which make claims about the level of students' understanding, the extent to which their ideas are normative, or the conceptual structures that they might "possess." We are particularly impressed with the analytic power of the completion process, allowing us to articulate the tools by which students elaborate their blends. Thus, our analyses lie fully within the domain of enacting ideas.

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