Themes in Undergraduate Students' Conceptions of Central Angle and Inscribed Angle

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Researchers have investigated students' multifaceted conceptions of angle and their difficulties with connecting angle measure to arcs or circles. In this study, we investigated three undergraduate students' thinking about angles in the context of circle geometry, specifically their conceptions of central and inscribed angle. Conceptual analysis of the data revealed that students involved in the tasks and interviews had various conceptions of these angles that either supported or constrained their ability to complete the tasks. Particularly, conceiving the dynamic transformation of both central and inscribed angles, or identifying their common subtended arc was productive, while considering angle as area or ray pair constrained their thinking.

Keywords: Student Thinking, Angle Conception, Geometry, Preservice Secondary Teachers

The Common Core State Standards for Mathematics (CCSSI, 2010) covers angle content in Grade 2 through High School, starting from identification of angles in planar shapes to angles in trigonometry. These topics highlight the complexity and variety of angle meanings in school mathematics, including angles as "geometric shapes that are formed wherever two rays share a common endpoint," angle measure with reference to a fraction of a circle, angle measure as a turn, and the relationships between central, inscribed, and circumscribed angle (ibid).

Despite the efforts studying students' understandings of angle (Keiser, Klee, & Fitch, 2003; Mitchelmore & White, 1998; Moore, 2013), little attention in mathematics educational research has been given to students' conceptions of angle involved in circle geometry. In the present study, we attempt to gain insights into students' understandings of angle by exploring three undergraduate students' ways of reasoning as they are asked to identify a central angle corresponding to a given inscribed angle in a circle. We illustrate the students' multiple conceptions of angle and how their different ways of thinking affect their ability to identify a corresponding central angle. We conclude by discussing what approaches potentially promote students' understandings of the relationship between central and inscribed angle and instructional implications.

Background and Motivation

Researchers have discussed how mathematicians in history (Keiser, 2004; Matos, 1990, 1991) and students and teachers (Clements & Battista, 1989; Keiser et al., 2003; Krainer, 1993; Mitchelmore & White, 1998) conceptualized angle concept. There are three viewpoints of angle that occur repeatedly in this literature: (1) angle as ray pair, (2) angle as region, and (3) angle as turn.

Students who conceive angle as ray pair construct an image of angle formed by two rays meeting at a common vertex. Mitchelmore and White (1998) indicated nearly forty-five percent Grade 4 children's responses of their angle definitions reflected they conceptualized angle in a way similar to this. Some third-grade and sixth-grade students' definitions of angle included: "an angle is where two vertices meet and make a point," or "[i]t's when two lines meet each other and they come from two different ways" (Clements & Battista, 1989; Keiser et al., 2003). Students who conceive angle as region consider an angle as a space bounded a ray pair. In this construction, a ray pair will contain two angles (large and small) instead of being one angle itself

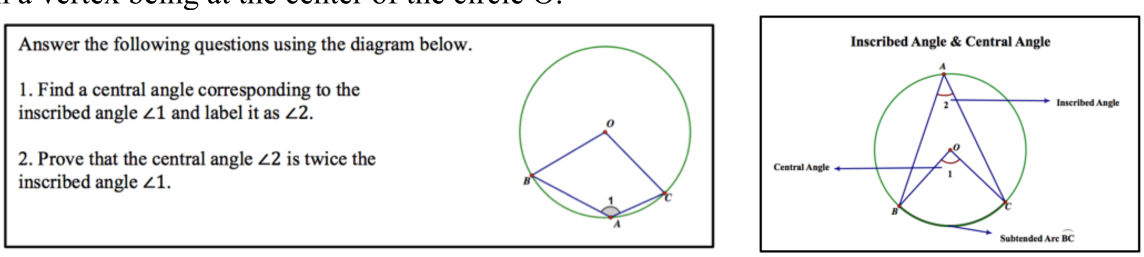
(Krainer, 1993). Fifteen percent of the four-graders' in Mitchelmore and White's (1998) study defined angle as an area. Students who conceptualize angle as turn or opening consider an angle as being formed by a dynamic rotation of one ray from another or angle as describing such rotation. Mitchelmore and White (1998) indicated only 4 out of 36 elementary students interviewed defined angle as turning. Clements and Battista (1989) suggested third graders who had Logo experience were more likely to define angle as a certain amount of rotation. Some children in this study defined angle as "something that turns, different ways to turn," or "when you turn some degrees" (Clements & Battista, 1989).

Noticing that researchers of most of these studies have focused on elementary students' *concept definitions* (i.e., words used to specify a concept) of angle, we consider it necessary to draw attention to their *concept images* (i.e., the total cognitive structure that is associated with a concept, which includes all the developing mental pictures and associated properties (Tall & Vinner, 1981; Vinner, 1991)). The focus of our study is to identify these concept images of angle. Specifically, given the paucity of research on undergraduate and/or teachers' conceptions of angle, our study aims at answering two research questions:

1. What are undergraduate pre-service teachers' concept images of central and inscribed angle in the context of circle geometry?
2. In what ways do these conceptions support and/or constrain their ability to solve circle geometry tasks?

Methods

We investigated the mathematical thinking of nine undergraduate students majoring in secondary math education from a large public university in the United States. The study consisted of three tasks: a pre-test, a reading task, and a post-test followed by a short interview. Each student completed the series of tasks individually. In the pre-/post-test, students were asked to complete a proof (Figure 1a) with the help of a handout that included a graphical definition of central and inscribed angle (Figure 1b). The normative solution of Question 1 is the reflex angle with a vertex being at the center of the circle O.



(a) (b)

Figure 1. (a) Pre-/post-test problem, (b) Inscribed and central angle definition handout.

After finishing the pre-test, they worked on the reading task set up on a computer. We designed two sets of presentations for this task (i.e., Static and Dynamic), and randomly assigned students to them. The Static presentation demonstrated the proof of the Inscribed Angle Theorem in three different cases and the supporting diagram of each case is static (see Figure 2a-c). The Dynamic presentation contained a dynamic diagram with a slider (see Figure 2a-d) The slider allowed students to move the Point C along the circle so that infinite cases could be seen. Meanwhile, the proofs would appear to the right of the figure depending on which static case the current state of the diagram belonged to. The fourth static case: to be tested in the post-test, was omitted from the static presentation and left blank in the dynamic presentation (Figure 2d).

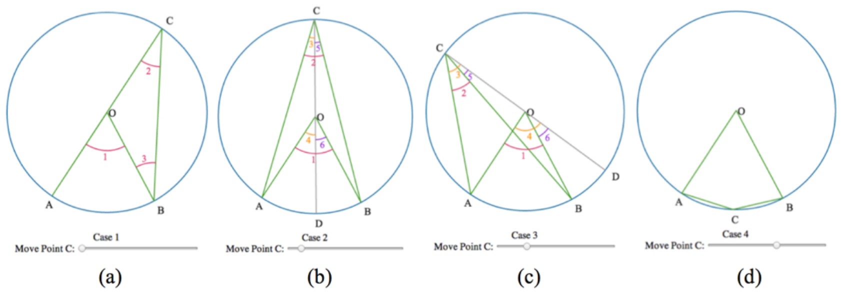


Figure 2. Snapshots for the four cases in the Dynamic presentation. The Static presentation included Case 1-3 without the slider.

After the post-test, each student participated in a clinical interview (Clement, 2000) to reflect on their thinking of the three tasks. We audiotaped all interviews and digitized students' written work. The process of how students drew the diagrams and thought aloud was also recorded with a Smart Pen throughout the study. Upon completion of data collection, we transcribed the interviews and incorporated figures and annotations.

In data analysis, we conducted conceptual analysis of an individual (Thompson, 2008) in order to develop models of a student's mathematical thinking. As researchers, although we cannot have access to students' minds, it is possible for us to make inferences about their mathematical thinking in ways that are consistent with our interpretations of their talking and observable actions. Both ongoing and retrospective analysis involved constructing conceptual models of pre-service teachers' meanings of central and inscribed angle. Using the observed conclusions about central and inscribed angle the students reached, we constructed hypothetical mental operations that would viably justify those conclusions that comprised these models.

Due to space constraints, we only report the reasoning of three of the nine students: Joanna (Static group), Hayley (Static group), and Jack (Dynamic group). We choose these three students because their stories establish the existence of highly varied understandings of angle among undergraduate pre-service teachers and the mathematical consequences of those understandings.

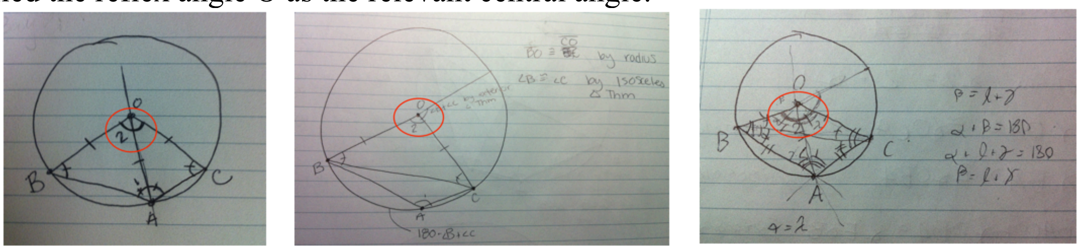
Results

We organize the three students' conceptions of central and inscribed angle into five themes (Table 1) and describe how these conceptions influence their ability to identify a central angle.

Table 1. Themes in students' conceptions of central and inscribed angle.

Theme	Theme Description	Students
Outside-Inside	A student considers an inscribed angle as being inside an area bounded by a quadrilateral (within a circle) while a central angle as being outside the same quadrilateral.	Joanna
Angle Size as Area	A student considers the openness (as area) of a central angle should be bigger than that of an inscribed angle.	Joanna
Angle as Ray Pair	A student considers a central angle as the minor angle constructed by two radii meeting at the center of a circle.	Hayley
Shared Arc	A student considers an inscribed angle and a central angle should share a subtended arc.	Hayley
Oriented Angle Rotation	A student considers the orientation of an angle as starting from one ray and ending on the other; the orientation of central and inscribed angle should be consistent.	Jack

In the pre-test, all the three students identified the smaller (obtuse) measure of angle O as the angle measure relevant to Question 1 (see "∠2" in Figure 3a-c). In this section, we will report our analysis of the post-test and interviews, where the three students changed their minds and identified the reflex angle O as the relevant central angle.



(a) (b) (c)

Figure 3. Diagrams produced by (a) Joanna, (b) Hayley, and (c) Jack during the pre-test.

Outside-Inside: Joanna

When working on the post-test problem, Joanna first realized that the central angle she labeled in the pre-test might be incorrect:

Joanna: …But that ["∠2" in Figure 3a] was not the central angle so then I didn't know what to do. So then I got a little bit confused.

Int: So you have discovered that this is not the central angle?

Joanna: Yeah, I figured it out, because…well because one has to be outside, one has to be inside. And they are both inside, so then I figured out that can't be right…

Joanna claimed that, for the central angle and the inscribed angle, "one has to be outside, one has to be inside." By comparing her own drawing (Figure 3a) and the figure in the handout (Figure 1b), Joanna realized that the central angle she labeled was incorrect, since she thought the area bounded by the correct central angle should not be also "inside" the quadrilateral ABOC. Joanna's "outside-inside" conception can be thought of as relative to the area of the quadrilateral in the circle (see the quadrilateral shaded in orange in Figure 4), with the central angle being outside and inscribed angle inside. We infer that she was conceiving angle as area and the relative positions of the angle-areas of central and inscribed angle. Eventually, Joanna changed her solution and labeled the reflex angle as the central angle.

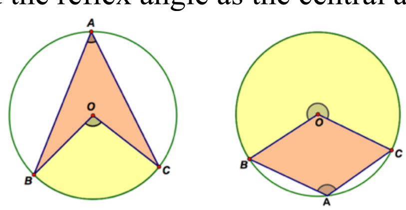


Figure 4. Interpretation of Joanna's "outside-inside"conception of central and instribed angle.

Angle Size as Area: Joanna

Although Joanna correctly (from our perspective) chose a central angle, she was uncertain about whether the angle was correct, saying "I figured it out but I don't know whether that was right, because I don't think it is." She thought the angle she labeled could not be right because "it was like the way too big to be a central angle, I don't think that's a central angle". Joanna had difficulty with conceiving an angle that is greater than 180°. As Joanna was conceiving angle-areas, here we inferred, Joanna's "too big" (the size of angle) probably referred to the measure of the openness of angles as the areas enclosed by the angles. The reflex nature of the angle might have made it appear to Joanna that the angle was enclosed by the area, rather than the reverse.

Later, she provided an explanation of what confirmed her choice of that central angle. She claimed that the openness of a central angle should "be like…really big" and that it "might be even bigger" than the given obtuse inscribed angle. This idea – a consequence of the theorem she was trying to prove – gave her enough confidence with her selection of central angle to finish the task.

Collectively, Joanna was reasoning with the position (i.e., her "outside-inside" approach) and the size of the angle-areas (i.e. "might be even bigger"). We consider her conception of angle as area in general as the fundamental reasons for her uncertainty about the correctness of her central angle. She continued to use hedge-words in the interview, and despite her success of providing a proof of the Inscribed Angle Theorem, she was not convinced that she had found a correct central angle.

Angle as Ray Pair: Hayley

In the interview, we started with asking Hayley about her reasoning in the pre-test:

Int: How do you think about this problem? Where did you get stuck?

Hayley: Umm…the arc part, like finding the first angle…this was the central angle ["∠2" in Figure 3b], right?

Int: What do you think is a central angle?

Hayley: The central angle would be in the middle of the circle [moving her hands along the two rays towards the center] cause this is the center, so that's why I put that this "O" is the central angle.

Hayley was describing a central angle as an angle constructed by two radii meeting at the center of a circle. She only conceived of the two radii BO and CO as constructing a single minor angle. Hence, she could only build a correspondence from this singular angle to the subtended minor arc BC and considered angle "O" to have exactly one measure.

Hayley also had a difficulty building a correspondence between central and inscribed angles, instead looking at each angle individually:

Int: Do you think going through these three cases would help you identify the central angle?

Hayley: I don't know because it is in the same spot. In all those and they are always the same angle in all them ["∠1" in Figure 2a-c]…because they are the same angle, so…yeah, I don't think that will be helpful.

Her awareness of the invariance of the angle location (i.e., "in the same spot") across cases suggested that she was considering the location of a central angle was absolute rather than relative to the inscribed angle. Due to the central angles of the first three cases being less than 180 degrees and thus fitting into her minor angle conception, the presentation was not helpful for Haley to change her previously identified central angle to the reflex angle, and thus she went with the same central angle as a solution for the post-test.

Shared Arc: Hayley

Haley's difficulty stemmed from her approach of first identifying a central angle and identifying the arc corresponding to that angle. In the subsequent interview, the interviewer instructed her to instead identify the subtended arc of an inscribed angle first, and then to find the corresponding central angle of that arc. The interviewer and Hayley went through all the three cases using this approach to identify corresponding central angles with given inscribed angles. When Hayley looked at the post-test problem again, she identified the correct central angle for the first time by making use of the shared subtended arc of the central and inscribed angle.

Oriented Angle Rotation: Jack

In the interview, Jack talked about how he identified the correct central angle by interacting with the dynamic diagrams in the reading task. When looking at the presentation, Jack carefully tracked the angles as the point C moved along the circle and paid particular attention to the transition between Case 3 (Figure 2c) and Case 4 (Figure 2d). He interpreted this process as "take[ing] a limit," by which he meant he was trying to exhaust all the details in between Case 3 and Case 4 to carefully observe how the angles changed and how they were "opened up" differently in this process. He later explained what changed between Case 3 and Case 4 in terms of the angles that made him refined his original choice of the central angle:

> "… you like keep track of the angles as they move because you can see here [Figure 2c], you know these angles stay the same, the same, but they just flipped over [Figure 2d], so you can just sort of generalize it."

Jack was interpreting when Point C went through Point A (from Case 3 to Case 4), the inscribed angle changed its orientation ("flipped over"; Figure 5). Jack may have imagined one ray to be the starting ray (AC), and the other ray to be the ending ray (BC). So as the orientation of the angle flipped (Figure 5; left to right), the direction of rotation also changed (counter clockwise to clockwise). Therefore, the original central angle AOB constructed by AO as the starting ray and BO as the ending ray should also change to the reflex angle AOB constructed by the same starting and ending rays but rotating from a clockwise orientation instead. Another interpretation of Jack's "flipped over" is that before C passes A, the inscribed angle ACB is constructed by AC on the angle's left and BC on the right (facing into angle C from the bottom of circle). After C passes A, the angle is constructed by BC on the angle's left and AC on the right (facing into angle C from the top of circle). So an inscribed angle and its "flipped" angle were orientationally different, and thus the original central angle AOB should also "flip" to its reflex angle with BO on the left and AO on the right (viewing angle C from the top of circle). Both interpretations lead to the same mathematical conclusion, so we consider them equivalent.

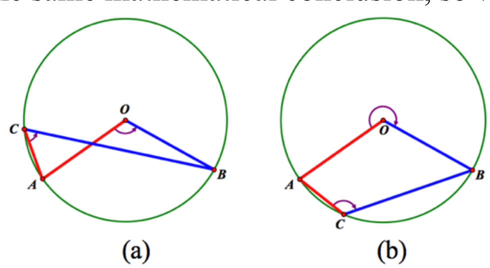


Figure 5. Interpretation of "Jack"'s "flipped over" as the orientation of an angle changing from (a) counter clockwise to (b) clockwise rotation of one ray from another.

Conclusions

The results of our study indicate that, regardless of angle contexts or grade levels, students' understandings of angle as ray pair (i.e., Hayley), angle as rotation (i.e., Jack), and angle as area (i.e., Joanna) persist from elementary students (Clements & Battista, 1989; Foxman & Ruddock, 1984; Keiser et al., 2003; Mitchelmore & White, 1998) to late undergraduates. That these undergraduates' conceptions of angle are similar to the definitions elementary students learn in school should not be surprising. What should be considered significant, however, is the impoverished nature of these images. These advanced undergraduates, many of whom will become mathematics teachers, do not have understandings of angle that have advanced very far beyond ray-pair, rotation, and area. Consequently, all the students struggle to track a changing inscribed angle and thus have difficulties in finding its corresponding central angle.

Although there is a very large body of work on identifying student definitions of angle (e.g., Keiser et al., 2003; Mitchelmore & White, 1998), there has been little work done on identifying students' concept images of angle *and* the mathematical consequences of these images. We have not merely identified Joanna as having an area-meaning, Haley as having a ray-pair-meaning, and Jack as having an orientation-meaning (possibly a rotation-meaning) of angle. We have also shown that these meanings directly cause the students' struggles and successes. Joanna, who conceived angle measure as area had difficulties with conceiving angles greater than 180° since these angles are "too big" to enclose an area. Additionally, her "outside-inside" approach is too specific to the particular situation to be a generalizable understanding of angle. It will easily fail in the situations where no reference objects or shapes (i.e., the closed figure: the quadrilateral) can be identified, or where it is necessary to conceive of a single angle that changes between "inside" and "outside" among cases. The case of Hayley suggested that students who had a ray-pair conception may not inherently or easily conceive of the structure of two segments as having two measures, and therefore constructing two angles. Only perceiving the minor angle constructed by two segments potentially results in students' difficulties with conceiving angles of 0°, 180°, 360°, and larger than 360° (Keiser, 2004). Fortunately, Haley's conception of angle measure as arc provided her a foundation to perceive the shared subtended arc, which was critical in her success. Finally, Jack's image of an angle as having an orientation (or a rotation) supported him to correctly keep track of the inscribed angle in the dynamic situation.

Contributing to the previous work on classification of students' angle conceptions (Clements & Battista, 1989; Keiser et al., 2003; Krainer, 1993), these three students' profiles indicate how the multiple meanings of individual students can interact. Their concept images of angle may not simply be "angles are areas" or "angles are arcs." For instance, Haley's difficulty with perceiving the major arc is generated by the complex combination of angle as a ray pair and angle as arc (i.e., one ray pair only corresponds to one angle measured by one arc).

Lastly, our findings highlight the need for supporting student understandings of angle in the context of circle and arc. The presence of the circle context of the tasks did not inherently lead the students to incorporate circles and arcs into their identification of central angle. Ultimately making use of the circle context was critical to the success of both Hayley (who found a common subtended arc) and Jack (who imagined the angle orientation changing as the vertex moved around the circle). Despite our attempt to assist Joanna to identify the subtended arc shared by an inscribed and central angle, she did not consider this approach as useful since she did not imagine the arc of a circle as having a role in angle measure. We hypothesize that if these students had an image of angle measure as arc (Moore, 2013), they would be more comfortable with relating central and inscribed angles using their shared arcs.

In order for teachers and researchers to be better able to recognize, explain, and respond to student thinking, and identify ways to assist them, we need to further explore their various conceptions of angle and angle measure, attend to the nuances of student thinking and its mathematical consequences, and be sensitive to the students' awareness (or lack of awareness) of the relationships between ray-pair, area, angle measure, circle, and arc.

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