Development of the Inquiry-Oriented Instructional Measure

George Kuster[1], Rachel Rupnow[2], Estrella Johnson[2], Annie Garrison Wilhelm[3]

[1]Christopher Newport University, Newport News VA.
[2]Virginia Tech, Blacksburg VA.
[3]Southern Methodist University, Dallas TX.

In this article we discuss the Inquiry Oriented Instructional Measure (IOIM). The development of the IOIM was a multi-phase, iterative process that required analyzing current research literature and videos of classroom instruction and piloting the measure with both experts and novices. The process resulted in identifying multiple instructional practices that support the successful implementation of Inquiry-Oriented Instruction (IOI) at the undergraduate level, and creating a rubric for evaluating the degree to which one's classroom instruction is reflective of these practices. Our goals with this paper are to share the development process and elaborate on the rubric so as to contribute to the knowledge base regarding the implementation of IOI.

Keywords: Inquiry-oriented instruction, Instructional measure, Teaching

Student-centered forms of instruction have been shown to have many positive outcomes for undergraduate mathematics students. Empirical studies demonstrate that Inquiry Based Learning (IBL) is a more equitable form of instruction and leads to greater affective and cognitive gains when compared to non-IBL teaching methods (Laursen, Hassi, Kogan & Weston, 2014; Kogan & Laursen, 2014). These outcomes directly align with the goals of recent calls for improving undergraduate STEM education. Ferreni-Mundy and Gucler (2009) noted all of the calls for reform in STEM education surrounded increasing student understanding of concepts, providing equitable access to students, and transitioning away from traditional teaching approaches to those that are student-centered and involve strategies that encourage active learning. With the push to increase the quality of STEM education, implementing such forms of instruction is important.

Instructional measures are one tool that can be used by various groups within the community to support the successful reform of undergraduate education. Researchers can utilize measures to assess the effects of instructional interventions, and practitioners can use measures to improve their instruction. In addition, measures can provide a vernacular and specific descriptions of instructional practices that promote instructional change. In this paper we begin with a brief discussion of a National Science Foundation funded project, *Teaching Inquiry-oriented Instruction: Establishing Supports* (TIMES). We outline the general design and provide a detailed account of the development of the inquiry-oriented instructional measure (IOIM), a measure for evaluating the degree to which a lesson consists of practices that reflect inquiry-oriented instruction (IOI).

Background

This work stems from the TIMES project, the goal of which was to scale up inquiry-oriented curricular materials (including developing instructor materials) for Abstract Algebra (Larsen, Johnson, & Weber, 2013), Differential Equations (Rasmussen, 2007), and Linear Algebra (Wawro, Rasmussen, Zandieh & Andrews-Larson, 2015). These curricula are research based and have been continually refined over the past two decades to scaffold student reinvention of mathematical concepts. The grant led to widespread dissemination and implementation of the curricula by recruiting mathematics instructors from across the United States. These instructors participated in activities intended to support them in implementing instruction that aligned with the four underlying instructional principles of IOI (Kuster, Johnson, Keene & Andrews-Larsen, 2017). The IOIM was developed as part of this project to help evaluate the efficacy of the support activities.

Generally, evaluation tools used for research and practice serve specific purposes; purposes that align with the goals of the research being performed. Common observation protocols and instructional measures include the instructional quality assessment (IQA), the mathematics quality of instruction (MQI), and the reformed teaching observation protocol (RTOP). The IQA was developed with a focus on "opportunities for students to engage in cognitively challenging mathematical work and thinking" (Boston, Bostic, Lesseig & Sherman, 2015, p. 160) and revolves around assessment of cognitive demand (Boston, 2014). The MQI was developed to aid in drawing connections between teacher knowledge and classroom instruction (Hill et al., 2008) and focuses on evaluating the quality of the mathematics available to students during instruction. The main goal of the RTOP was to serve as a tool for pedagogical development aimed at improving instruction. The RTOP is designed to measure the degree to which classroom instruction is *reform-oriented* (Sawada et al., 2002). With regard to the TIMES project, we found that these tools and others like them did not fit the specific needs of the project, in that they failed to attend to the nature of IOI to the degree we needed. Our goal was to focus on the instructional practices in which the teacher engaged while in the classroom.

Overview of the Inquiry-Oriented Instructional Measure

The IOIM is a rubric designed to provide quantitative and descriptive data concerning the enactment of the four main instructional principles of IOI: *generating student ways of reasoning, building on student contributions, developing a shared understanding, and connecting to standard mathematical language and notation* (Kuster et al., 2017). Broadly speaking these principles reflect three characteristics of the role of an IOI teacher: 1) inquiring into student mathematics, both in terms of individual students and in terms of the learning trajectory (Rasmussen & Kwon, 2007; Johnson & Larsen, 2012); 2) being an active participant with the developing mathematics, both in terms of the mathematics of the moment and in terms of the mathematical trajectory intended by the curricular materials (Johnson, 2013; Johnson & Larsen, 2013); and 3) bridging the gap between where the students are and the mathematical goals of the lesson (Wagner, Speer & Rossa, 2007; Speer & Wagner, 2009). The IOIM consists of a set of practices that support the enactment of each of the principles, and a rubric (shown at the end of this report) that measures the degree to which a lesson is inquiry-oriented by examining the quality of the enactment of these practices.

Each of the practices is scored on a 5 point likert-scale from low to high. Generally, within each of the practices, the quality of the mathematical activity promoted by the teacher is what distinguishes a low (1) from a high (5). Take for example Practice 2: teachers elicit student thinking and reasoning. If a teacher evokes solely procedural contributions from students, they score significantly lower (medium-low) than if they routinely have students share their thinking, reasoning and justifications (high).

Development of the Inquiry-Oriented Instructional Measure

The measure was developed in five phases using data consisting of research literature, videos of classroom instruction from both expert and novice IOI instructors, expert validity checks, and notes from pilot training-sessions. The overall process began with codes and categories that were developed from the data and iteratively refined in the process of creating a descriptive framework of IOI. In addition, new data was sought to specifically address questions and hypotheses as they arose, and subsequently led to the refinement of the framework. Other research methods were incorporated into the phases such as Lesh and Lehrer's (2000) iterative video analysis. In the following sections we outline the work completed in each phase.

Phase 1: Defining the Task - What is IOI and how do we measure it?

This phase resulted in a general understanding and vocabulary for characterizing IOI and information on how to measure teaching in general. In this phase, we searched through research literature for defining characteristics of IOI and determined that a distinguishing feature was that the teacher, students, and tasks each have a critical and active role in developing the mathematics. After returning to the literature and coding for "teacher", "students", and "tasks", we generated a starter list of instructional practices of IOI and supported these practices with justifications and examples from the literature. After our working definition of IOI was complete, we examined existing instructional measures (e.g., RTOP, IQA, MQI) to determine if they adequately captured our characterization of IOI. Though it was ultimately determined the other measures were not applicable, they did influence the refinement of practices and provided useful descriptions for what some practices looked like when enacted.

Phase 2: Examining Data - Verifying practices and identifying measure limitations

Although the research literature led to the identification of numerous practices of IOI, the utilization of a measure required being able to *observe* these practices. In this phase, we cycled between analyzing videos and existing literature to verify that the practices identified from the literature were also evident in classroom instruction. In the first pass through the video data, we watched two expert instructors (IO curriculum developers) and three novices. The variation in experience level was purposeful; we intended it to highlight key aspects of instruction. While watching these videos, we documented the classroom events with content logs, coded for critical components, and wrote narratives for each of the practices based on what was observable in the videos. This process resulted in refining the list of practices and their characterizations. Thus, the characterizations of the practices were created in terms of supporting literature and video data.

Once the practices were defined and descriptions of their enactment were created, we delineated across the various levels (i.e., high, medium and low) at which instructors performed each of the critical components. Using the video data, we created a rubric for scoring the quality of the implementation of each component by ranking the various instructors in terms of how well their instructional practices aligned with the tenets of IOI. We then identified themes within the various levels of quality by comparing across the components within each of the scores. The process of ranking the instructors also raised important questions regarding issues such as how these practices connected to each other and how they fit within the four instructional principles.

Phase 3: Refinement using outside sources

In this stage, we began seeking resources from beyond IOI research literature and feedback from researchers not directly involved in the development of the measure. First, we asked a researcher not familiar with IOI to code two videos with the drafted rubric. After discussing areas of confusion and working out discrepancies between scores, we began searching through K-12 research literature looking for aspects of K-12 instruction that were commensurate with the practices we identified in IOI. These steps led to refining the practices and led to a better understanding of the principles and the supporting practices. Specifically, the K-12 literature was able to provide descriptions for what we noticed from the IOI video data and language for delineating among the various levels of implementation.

Phase 4: Sharing to clarify

In this phase our intent was to pilot the rubric with experts and novices to both clarify connections between the principles and practices and work toward a common interpretation of them. We first asked researchers familiar with IOI but not with the measure to use the rubric to score the same lesson. While this step had multiple benefits, there were two important outcomes. Most importantly, despite no training, the scores across all six researchers (including two rubric developers) were all within one point. Thus, while some reorganization was needed, the descriptions in the rubric were generally meaningful to researchers familiar with IOI.

We then engaged in a pilot training process where we trained three graduate students having no background in IOI on how to use the rubric. During this process we asked the coders to take careful notes of issues that arose for them as they utilized the rubric. We also recorded the meetings when we met to discuss the scores they assigned. From this we concluded that two practices were capturing the same aspects of instruction and removed one of them. We also created resources for coders, including guiding questions, “evidenced by” descriptions, and boldfacing certain words in the rubric.

Phase 5: Sharing to use

In this phase, we implemented the full scale training of six graduate students from various mathematics education backgrounds. Training started with having the coders watch video clips exemplifying the different levels of IOI for each of the practices. As training progressed, coders were given more opportunities to watch longer segments of classroom video with a partner or on their own each evening and to justify their own scores using the rubric. In group meetings, coders would then engage in facilitated debates of their scores, which allowed misunderstandings of terms and weaknesses in justifications to be resolved. The coders were also encouraged to articulate in their own words what each practice would look like at high, medium, and low levels as another check of their understanding.

At the end of a week of training, coders were given a test video to determine their readiness to code independently. All coders gave scores within 1 level of the trainer's scores, which allowed them to be released to code videos gathered from TIMES instructors. Five of the six coders then went on to each score eight to twenty-one other videos. (The sixth did not score any videos after training.) In order to insure reliability, the trainer had a meeting with each coder after every fifth video to make sure all scores remained within 1 of the trainer's scores. In seven of the ten meetings, the scores were all within 1 of the trainer. In cases where the coder was off by two levels, they were asked to rewatch and rescore the video in light of the discussion with the trainer before being allowed to continue scoring videos.

Discussion

In this paper, we outlined our development of a rubric for IOI. Creating a measure for IOI at the undergraduate level presented non-trivial and unique challenges. First, it was necessary for the IOIM to have the flexibility to be utilized across an array of undergraduate mathematics courses. Though, not only does the content differ across introductory courses such as differential equations, linear algebra, and abstract algebra, most notably, the mathematical goals are often vastly different. For instance, an introductory differential equations course is often intended to develop an understanding of solution methods, whereas introductory abstract algebra is often utilized to develop notions of formal mathematical proof. Instead of being overlooked this difference in goals needed to be flexibly built into the measure.

Second, the IOIM needed to incorporate a wide variety of instructional strategies. From a theoretical standpoint, in IOI the teacher navigates along the continuum of pure telling and pure student exploration (Rasmussen & Marrongelle, 2006). From a practical standpoint, flexibility across instructional types was necessary because of the nature of the TIMES project: supporting instructional change. That is, the measure needed to provide information regarding how the participating instructors were incorporating aspects of IOI into their instruction and to what degree they were doing so. These challenges and others greatly influenced the resulting structure of the measure. With this we hope to contribute to a broad community, one consisting of mathematics education researchers as well as practitioners.

Inquiry-Oriented Instructional Measure

Practice	Component	Description	5	4	3	2	1
			High	Medium High	Medium	Medium Low	Low
1. Teachers facilitate student engagement in meaningful tasks and mathematical activity related to an important mathematical point.	Generate	This practice is characterized by student engagement in mathematical activity related to an important mathematical point. This activity is likely to be supported with meaningful, cognitively demanding tasks. (Jackson et. al, 2013; Hiebert, 1997; Speer & Wagner, 2009). However, we are not evaluating the tasks, rather the quality of the mathematical activity. Stein's "doing mathematics" includes conjecturing, argumentation ... all of the things we call engaging in authentic mathematical activity. (Stein, et al., 2008) This activity is likely to be supported by providing the students with meaningful, cognitively demanding tasks and supporting their engagement in those tasks through questions that promote the identification and investigation of the mathematical concepts.	Students were engaged in generalizing their thinking to mathematical claims that they then tested/supported with arguments, examples, and counterexamples. The teacher tried to connect the task to the students' prior experiences with guidance that helped facilitate continued engagement in the task. The students were "doing mathematics." **"Doing Mathematics"**	While the students were provided opportunities to "do mathematics" the majority of the mathematics during the discussion was performed by the teacher. The teacher used the tasks to engage the students in the conceptual ideas. **"Procedures with connections"**	The teacher engaged students in the tasks, focusing on the correct answer or the procedures as opposed to the mathematical ideas underlying those procedures. **"Procedures without connections"**	The students engaged in few mathematical practices. The majority of the "learning time" consisted of watching the instructor provide the correct answer. The teacher did not utilize the tasks to support the development of the mathematics. The teacher engaged students in tasks that largely focus on memorization of important mathematical ideas. **"Memorization"**	The teacher lectured the material (i.e. there are no mathematical tasks for the students to engage in). The majority of the mathematical activity was carried out by the teacher, with little to no time for students to do anything but watch. **No space for student mathematics**
2. Teachers elicit student reasoning and contributions	Generate	Teachers prompt students to explain their reasoning and justify their solution strategies, with the focus on the reasoning the students utilized during the task as opposed to solely focusing on the procedures used. Research on instructional quality indicates that the type of contributions teachers elicit is directly related to the students' opportunities to learn. Thus it is important that teachers elicit thinking and reasoning that "uncover the mathematical thinking behind the answers" (Hufferd-Ackles, Fuson & Sherin, 2004, p.92).	The teacher explicitly asked students to share their approaches to the tasks and their reasoning. The focus was not on what answer they got, but more on how they arrived at that answer, how they were thinking about the problem, the method they took, and a justification for choosing that method. Students may be sharing **thinking/reasoning** and questions without prompting from the teacher. Teacher questions and student contributions are about student reasoning, methods, and justifications, with students providing full and meaningful contributions.	The teacher routinely and explicitly asked the students to share contributions pertaining to their thinking and reasoning such as, "Why do you think the examples you generated are working that way?" "How did you do it?" and "What does this mean?" The teacher, however seemed to **interpret and revoice these contributions** for the students without allowing the students to fully explain them or respond to them, which took ownership of the ideas away from the students. Although the student's line of reasoning on the solution path is given, why the student chose that solution path is not.	The teacher typically elicited descriptions about how students solved a task, but not about how they were thinking about and/or approaching the task; **solutions and descriptions of how those solutions were obtained are the focus.** Additionally the instructor often talked about the students' contributions as opposed to allowing them to explain their thinking. The instructor elicited solutions but not reasoning. This includes well-worded questions that are asked in such rapid succession that they lose their ability to elicit students' reasoning. Alternatively, the focus of questions was still more on the correctness of procedures or answers, and not on students' understanding.	The instructor made an effort to elicit student contributions (e.g., reasoning, methods, and justifications) but often these attempts yielded **procedural statements**. Questions are segues into lecture.	The teacher asked fill in the blank questions. There were **little to no opportunities for students to make deep and meaningful contributions**. The Instructor does not ask students to share their thinking.
	Build	Leatham et al. [22] note that foundational to building on student thinking is the presence of student mathematical thinking that "is likely to advance students' development of important mathematical ideas—whether the student thinking is mathematically significant" (p. 92). Thus, in order to build on student thinking, rich contributions must be elicited.					
3. Teachers actively inquire into student thinking.	Generate	This practice means that instructors purposely and intently inquire into student thinking for the purposes of determining if and how student generated ideas can be utilized to promote a more sophisticated understanding of the mathematics. The questions asked by teachers not only direct student investigations and provide the teacher with insight into student thinking, they also help students refine and reflect on their own thought process (Borko, 2004; Hiebert & Wearne, 1993; Rasmussen & Kwon, 2007). By asking them questions about how they are thinking - you are supporting them in further engaging in their thinking.	The teacher engaged in conversations with the students about their thinking. These conversations indicate that the teacher was assessing their own understanding of students' thinking. There are instances of probing, rephrasing of student contributions, and representations. (Davis description of interpretive listening and Rasmussen, Kwon, Marrongelle's IO-DM, and Henningsen, Stein 1997)) It seems as though the teacher is trying to figure out, "How are my students thinking about this?" They are trying to **"build a model of students' understanding."**	The teacher appeared to **actively figure out the student's mathematical contributions** (e.g., reasoning, methods, and justifications) - were their conjectures valid, were the students contributions useful in developing the mathematical agenda? The teacher appeared to try to answer the question, "Are this student's mathematical **contributions useful?**", as opposed to "What sort of thinking led to this contribution?" *This is good pedagogical reasoning on the part of the teacher and very useful for responsiveness and connecting and all sorts of things - but it is not the main point of this practice.	The teacher attempted to figure out how the students solved a problem or task with attention to the concepts underlying that solution method. Their focus was on why **something worked**, not how the student was thinking about the mathematics. (E.g., "because of the associative property I can regroup these elements" provides a valid mathematical justification for how to do something but does not provide information about how the student was thinking about the task/mathematics.) **Thinking – procedures with connections**	The teacher appeared to be trying to figure out how the students solved a problem with attention to the **validity** of the student's solution **strategies**. That is the focus was on the correctness of the procedures, not on why those procedures worked. **Trying to figure out "Can they do it correctly?"**	The instructor's response to student contributions was limited to determining if the contribution (often just "the answer") matched a predetermined and expected answer. In this sense the teacher inquiry served an evaluative role, determining the **correctness of student responses**. There is no evidence that the teacher was trying to figure out how the students are thinking, just if their "answer" appears to be correct. Trying to figure out "do they have the right answer"
	Build	Teacher inquiry serves many functions and roles throughout a lesson (see Hufferd-Ackles, Fuson, & Sherin, 2004; Johnson, 2013; Rasmussen & Kwon, 2007). With regard to building on student contributions, teacher inquiry allows teachers to form models of student thinking and understanding, reconsider important mathematical ideas in light of those models, and formulate questions and tasks which enable the students to build on those ideas (Rasmussen & Kwon, 2007).					
4. Teachers are responsive to student contributions, using student contributions to inform the lesson.	Build	Rasmussen and Marrongelle (2006) state that, "an important part of mathematics teaching is responding to student activity, listening to student activity, notating student activity, learning from student activity, and so on" (p. 414). By doing so, the teacher can generate instructional space where "the nature of student mathematical thinking might compel one to take a particular path because of the opportunity it provides at that moment to build on that thinking to further student mathematical understanding" (Leatham et al., 2015, p. 118). Rasmussen: IO informs questions and tasks that enable students to build important math ideas.	The instructor listened to students' contributions (e.g., reasoning, methods, and justifications) and, when appropriate, used these ideas as a springboard for follow up questions and further exploration by the students. Questions about student contributions were asked to the class. The teacher used intermediate student progress to re-direct and focus small group work towards the important mathematical ideas. The instructor often took advantage of student thinking that they felt promoted the mathematical agenda by shifting focus to a student generated justification. Willingness and space to follow/explore unexpected contributions - where the students are doing the exploration. **Lots of instructional space: students build on each other's contributions/questions/suggestions**	Student ideas guiding the teacher's exploration - The teacher was willing to follow/explore unexpected contributions (e.g., reasoning, methods, and justifications), and provided space to do so. However the teacher was doing the exploration, not the students. **Some instructional space: teacher builds on students' contributions/questions/suggestions**	Student contributions (often answers evaluated for correctness) were used by the teacher as a way to support a predetermined path. Little attention or instructional space is devoted to unexpected contributions (or to eliciting unexpected contributions). Additionally, the contributions were often utilized as a way for the instructor to lecture about the topic/idea at hand or lead the students to replying in a predetermined manner. **Predetermined contributions to support the predetermined path.** *One possible way of getting a score of 3 in this practice, is if a "lecturer" explains a procedure, students practice procedure, and the teacher uses their answers to indicate it is time to teach the next procedure, to do more practice problems, or to do an additional lecture based on how the students did on the procedure.	The teacher rarely considered students' contributions with regard to the mathematical agenda. The use of student contributions was limited to "fill in the blank" responses that indicate is it time to move forward. For example, the teacher lectures for awhile, then the students practice a procedure, and then the instructor explains the correct answer before repeating the process. If necessary, the teacher might lecture further if there were common misconceptions. **Limited instructional space**	The teacher was adhering to a pre-determined lesson plan. There was little to no instructional space for exploring student ideas or questions. The teacher's instruction did not make reference to student contributions (apart from answering clarification questions). Essentially the teacher lectured for the whole period. **Little to no instructional space**
	Shared	When teachers are responsive to student contributions they can create new instructional space (Johnson and Larsen, 2012). In regards to this component, the instructional space is created for the purpose of developing a shared understanding within the classroom community.					

Practice	Component	Description	5	4	3	2	1
			High	Medium High	Medium	Medium Low	Low
5. The teacher engages students in one another's thinking (reasoning?)	Shared	Stein and colleagues (2008) provide several examples of how teachers can support students in making mathematical connections between differing student contributions and important mathematical ideas. Some of these examples include asking students to reflect on the contributions of other students, assisting students in drawing connections between the mathematics present in solution strategies and the various representations that may be utilized, and facilitating mathematical discussions about different student approaches for solving a particular problem. Doing so can prompt students to reflect on other students' ideas while evaluating and revising their own (Brendefur & Frykholm, 2000; Engle & Conant, 2002). By engaging with one another's thinking, students are able to deepen their thinking, generate new ideas, and make mathematical connections. As discussed by Jackson et al. (2013), "the teacher plays a crucial role in mediating the communication between students to help them understand each other's explanations" (p. 648).	During whole class discussions, the teacher makes instructional space for **students to reflect on the contributions of other students** - e.g., "How are you guys making sense of what they are starting to think about here?" The teacher encouraged students to ask each other questions about alternate approaches and ways of thinking. Additionally the instructor often pointed out ideas other students had while working on the task as a way to support the students' progress. The **teacher and students share responsibility for the mathematical ideas**. Essentially, students interact with each other's ideas without the teacher acting as much of a filter.	During whole class discussion, student ideas were made public and students were asked questions such as "Did anyone think of it a different way?" Most of the interactions left the students with **access to each other's approach but not to each other's thinking/reasoning or justifications**. In general, it did feel like the **student contributions were leading to meaningful class discussion**, as mediated or filtered through the teacher. While the students are responsible for the mathematics, **the teacher is the mathematical authority** in the room. Teacher rephrases/revoices student ideas for the purpose of having other students react/interact with it. This is done in a way that does not preserve the student's contribution - or alters/adds/"smooths" in a significant way. The students are interacting with the teacher's version of the student's ideas - as opposed to interacting directly with the other student's ideas.	Students contributed to whole class discussions, during which students were asked questions such as "Did people get the same/different answer?" However, students were not prompted to reflect/compare/assess other students' contributions or ideas or students' responses may not have included much reasoning. **Overall, there was not an explicit attempt to have students make sense of each other's contributions though they did have access to them. The teacher is still the mathematical authority.**	There were opportunities for students to engage in one another's thinking (e.g., small group work). However, **during whole class discussion and/or lecture, students did not have access to one another's thinking**. The whole class discussions seemed more like a space to contribute possible correct ideas for feedback from the instructor, as opposed to a space for students to engage in argumentation, explanation, and/or justification. For instance, if the only thinking elicited (Practice 2) is a med-low (i.e. procedural statements and answers are shared) then students cannot engage in one another's reasoning.	There is **no student reasoning** shared (in small group or whole group discussion) for the students to engage in.
6. Teachers guide and manage the development of the mathematical agenda	Build	Teachers need to actively guide and manage the mathematical agenda and can do so by: identifying and sequencing student solutions to "ensure that the discussion advances his or her instructional agenda" (Jackson et al., 2013, p. 648); utilizing Pedagogical Content Tools "to connect to student thinking while moving the mathematical agenda forward" (Rasmussen & Marrongelle, 2006, p. 389); or by refocusing the class towards the use of certain student generated ideas, marking important student contributions, and assigning tasks meant to clarify and build on students' ideas/questions. In these ways, teachers can guide and manage the development of the lesson while building on student contributions, developing mathematical ideas in directions commensurate with the mathematical agenda, and maintaining the student ownership of the mathematics.	The teacher monitored small group work, and both recorded and used student contributions during whole class discussions as a way to promote the development of the intended mathematical ideas. The teacher **guided student explorations towards the mathematical goals** of the day. This could include asking students questions to allow them to find a solution in their own way. The teacher's questions and interactions appeared to be directed towards a clear mathematical goal. Additionally the instructor managed both the small group work and the whole class discussions.	There was a **well defined structure within the class**, and the instructor focused the students on the important mathematical ideas. The instructor utilized strong guidance to direct the lesson consisting mostly of student contributions. Specifically the **instructor was responsible for exploring the mathematics, and the students were responsible for responding to that exploration**. This could include asking questions that lead students along the teacher's line of reasoning.	There was certainly a **well defined goal and structure** within the class. The instructor focused the students on the important mathematical ideas. The instructor however did **not** seem to be **utilizing student ideas to progress the mathematical agenda**.	Though there was a goal in mind, the **structure of the class was not well defined**. Additionally, the teacher did not facilitate the progression of the mathematical agenda.	The teacher did **not** appear to have a **clear mathematical goal** in mind. The students are left to explore the mathematics at their leisure.
7. Teachers support formalizing of student ideas/contributions and introduce language and notation after a need/understanding has been established.	Connecting	In inquiry-oriented instruction, as the students reinvent the mathematics, their reinventions build to be commensurate with formal mathematical ideas. The instructor must be able to promote the students' ability to connect their mathematical ideas to more formal mathematics. "The teacher plays a crucial role … in supporting students to link student-generated solution methods to disciplinary methods and important mathematical ideas" (Jackson et al., 2013, p. 648). Formal notation is introduced after the students have generated an understanding of what is being notated and a need for it has been established. "In contrast to more traditional teaching in which formal or conventional terminology is often the starting place for students' mathematical work, this teacher [one implementing an inquiry-oriented curriculum] chose to introduce the formal mathematical language only after the underlying idea had essentially been reinvented by the students" (Rasmussen, Zandieh, Wawro, 2009, p. 203).	The students **formalized their own ideas and contributions, doing all of the formalizing except being told that the concept is a standard one with a standard name**. The students were provided with opportunities to make connections between their work and the more formal mathematical notation (e.g., students are asked to translate and/or interpret formal mathematical nomenclature utilizing their own generated ideas). When appropriate, the teacher notated student contributions with formal mathematical language and notation. This was done after the students had a chance to informally develop the mathematical concepts, and when it was likely that the students could make sense of the formal mathematics by connecting it to their previous work. There is instructional space for students to formalize their own ideas - where formalizing can mean connecting to standard language and notation. This formalizing draws heavily on the students' work and contributions. There was instructional space for students to translate their contributions and prior reasoning into the newly introduced language and notation.	Teacher does the formalizing of ideas that originated with the students - where formalizing can mean connecting to standard language and notation. This formalizing draws heavily on the students' work and contributions. **The teacher used notation as a way to capture, notate and translate student ideas and contributions. The translation however was done by the teacher, not by the students.** The teacher formalized the students' mathematics and made the connections (i.e., there was not an opportunity for students to translate or formalize their own ideas - this was done by the teacher). The difference between this and high: with medium-high the teacher records student thinking with notation, but then the teacher does all the translating into the notation. With "high" the teacher introduces notation and then asks the students to do the translation.	Students were given an opportunity to work toward formalizing their own arguments and ideas. However, there is a **significant disconnect between what the teacher introduces and the students' work**. This can happen in a number of different ways. For instance: – The teacher does not draw on the students' ideas/work. Instead, the instructor follows up students' work with a summary that does not directly reference the students' ideas or makes no explicit connections between the students' ideas and the language and notation introduced by the teacher. In effect, the **teacher ignored the formalization students had done**. – The teacher does not attend to the ways in which the students are using formal language and notation, causing **a disconnect in the meaning of words used by the teacher and the students** (e.g., a student uses the term "inverse" or "one-to-one" and the teacher assumes a shared formal meaning). – There are **not enough student ideas** present for formalizing.	**The students were given a brief opportunity to formalize ideas, but they were not successful in formalizing.** This lack of success could be due to insufficient time to complete the task, a poorly framed task, or students not understanding well enough to approach the task. One example of this may be if the students were asked to complete a sequence of tasks, and at the end were asked to come up with a conjecture/generalization about a pattern common to task sequence. Their contributions were ill-formed or not utilized by the instructor. Additionally, there were little to no connections between the students' ideas that did arise and the language and notation introduced by the teacher.	**Formal mathematical language and notation were introduced prior to student engagement with a task.** The language and notation were introduced before a need had been established for it. (e.g., Definition/Theorem/Proof) **Alternatively, standard mathematical language and notation are not introduced to the students at any point in the lesson.**
	Shared	Mathematical language and notation does not only serve as a way to mark important mathematical ideas. In an inquiry-oriented classroom, the introduction of language and notation also serves as a way for teachers to support the use of a common way of **thinking** about a mathematical idea, which then serves as a launching point for the construction of more formal mathematics. Rasmussen and Marrongelle (2006) indicate that inquiry-oriented instructors often notate student thinking in ways that allow the class to then solve new problems.					

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