Professor Goals and Student Experiences in an IBL Real Analysis Course: A Case Study

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Through an in-depth case study of one real analysis course taught by a very experienced instructor, we gain insight about two goals expressed by advocates of Inquiry Based Learning (IBL) instruction: developing students' persistence in mathematical study and their identity as mathematics learners. The research study was guided by collaborative workshopping research priorities and questions with of a group of experienced IBL instructors. We provide an in-depth characterization of this highly-experienced instructor's conceptualization of his teaching practice in undergraduate Real Analysis; specifically, we identify how his deviation from conventional proof-oriented instruction served to uphold his key goals that students create proofs *and* overcome challenges*. We then use this characterization of his practice to report on students' experiences learning in the course, especially as related to the professor's two goals.*

Keywords: Inquiry-based learning, identity, persistence, instructor goals, motivation

Motivation for the Study and Research Questions

There currently exist multiple broad movements in undergraduate mathematics education toward various forms of inquiry-oriented instruction (e.g. Dawkins, 2014; Kogan & Laursen, 2014; Kuster, Johnson, Keene, & Andrews-Larson, 2017; Rasmussen & Kwon, 2007). In this context, the term *inquiry* covers a range of particular notions and functions. Kogan and Laursen's (2014) study distinguished Inquiry Based Learning (IBL) courses that spent more than 60% of class time on student-centered activities from non-IBL courses in which instructor speech occupied more than 85% of class time. We rather focus on particular values and goals endorsed by practitioners of inquiry-oriented instruction. This project was initiated by the authors' participation in a collaborative workshop hosted by the American Institute of Mathematics (http://aimath.org/pastworkshops/iblanalysisrep.pdf). The workshop brought together IBL real analysis instructors and mathematics education researchers focused on real analysis to foster professional partnerships and to outline some agendas for research on IBL real analysis instruction. One such research agenda focused on how IBL instruction influenced students' persistence in mathematical study and their identity as mathematics learners. In response, we formulated the current study of the teaching practice of one highly experienced IBL instructor (Professor X) and his students' experiences. We pursued the following questions:

1. What goals for student development does the IBL instructor articulate throughout teaching the course and reflecting on student progress?
2. How does the professor structure the course and his interactions with students to achieve his articulated goals and provide all students with appropriate opportunities to overcome the challenge of creating proof?
3. How do student interact with the course structure and instructor to navigate through the course and how do these trajectories achieve or challenge the instructor's learning goals?

Theoretical Perspectives on Inquiry in Mathematics Instruction

Rasmussen and Kwon (2007) provide an influential definition of inquiry for undergraduate mathematics instruction. In their view, classroom inquiry includes both 1) student inquiry into mathematical tasks that are meaningful and accessible to them and 2) the instructor's inquiry into students' mathematical reasoning. The first type of inquiry helps students see mathematics as a human activity in which they participate. The latter allows the professor to build instruction on student thinking. As we shall argue later, Professor X's practice was compatible with both criteria, though student inquiry was more prominent and Professor X built on student thinking in more indirect ways. Since it is rooted in Realistic Mathematics Education (Freudenthal, 1973; Gravemeijer, 1994) this tradition of inquiry emphasizes a range of mathematical activities such as defining, conjecturing, theoremizing, and proving. Professor X instead almost exclusively invited students to prove mathematical claims. He provided the vast majority of definitions and statements to be proven in the course script (though students were not always told whether the given statements were true or false). Kuster et al. (2017) describe four principles of this tradition of inquiry oriented instruction: generating student ways of reasoning, building on student contributions, developing a shared understanding, and connecting to standard mathematical language and notation. Of these, Professor X only focused on the first because he wanted to maintain the independence of student contributions in overcoming challenges.

Professor X is more directly aligned with the tradition of inquiry studied by Kogan and Laursen (2014). Professor X does not allow any collaboration among students in his course, which was a key component of the positive experiences Kogan and Laursen reported. Still, those authors go on to explain, "Public sharing and critique of student work may serve as vicarious experiences that enhance self-efficacy and link effort, rather than innate talent, to mathematical success" (p. 197). This explanation of how peer presentations may influence students' mathematical mindset (Good, Rattan, & Dweck, 2012) suggests affective mechanisms that would still be present in Professor X's class, though students worked independently.

Literature Review

Numerous studies attest to the significant challenges students face in learning the definitions and logic native to real analysis. Conceptual difficulties abound with limits (Oehrtman, Swinyard, & Martin, 2014; Pinto & Tall, 2002), monotonicity (e.g. Alcock & Simpson, 2017; Bardelle & Ferrari, 2011), cardinality (Shipman, 2012), completeness (Durand-Guerrier, 2017), and compactness (Dubinsky & Lewin, 1986). Definitions in analysis frequently include multiple quantifiers that evoke non-normative interpretations among students (Dubinsky & Yiparaki, 2000). Real analysis also includes proof methods students need to learn (Weber, 2001) such as universal generalization (Durand-Guerrier, 2008), absolute value inequalities, and constructing functional relationships between quantified terms. Students' must also reason fruitfully about examples and visual representations to coordinate their concept image with the concept definition (e.g. Alcock & Simpson, 2004; Tall & Vinner, 1981). As mentioned above, Professor X did not present on these definitions, but rather provided students with tasks to prove or disprove and expected students to learn about these concepts relatively independently. As we shall report, Professor X reorganized these concepts to facilitate student learning, though it is beyond the scope of this report to fully explore this conceptual reorganization.

Methods

The bulk of the research data gathered consisted of Professor X's reflections on the class meetings and student learning as facilitated by the two researchers. Two weeks into the class, after Professor X had time to get to know the students and allow them to settle into the routine of the course, we created an *Initial Summary Document* including summaries of the students based on Professor X's observations and past experience. Using categories created by the expert participants at the AIM workshop, we prompted Professor X to classify students' early proving capabilities, from Novice to Master. We then established two online collaborative documents, an *Interactive Diary* and a *Student Summaries*. In the interactive diary, Professor X recorded interesting and noteworthy episodes from the class as well as general comments about the course notes or classroom culture. Professor X also created notes on the work and progress of individual students in the Student Summaries file, based on student presentations, office hour visits, and homework. The researchers provided comments and questions in each document. We used different colors and date stamps for each entry to make it easy to track the dialogue among speakers and to notice what still required response. We also conducted monthly phone interviews to discuss the interactive diary and student summaries entries, adding to each as possible.

At the end of the semester, the researchers requested interviews from 15 students, five from each of three ranges of success at the end of the semester as perceived by Professor X. Three students from the lower success range (faux initials CR, TJ, and BT), four students from the middle range (LN, KC, SQ, and RU), and three students from the higher range (TC, OI, and OO) agreed to be interviewed. We crafted questions to elicit their perceptions of: 1) the course, 2) the IBL instructional methods, 3) professor and peer interactions in the class, 4) how they progressed in the class, 5) how they benefitted from the class, 6) what they struggled with, 7) what they were most proud of, and 8) what helped them most throughout the course. We conducted a 60-90 interview with each of the 10 participants with online videoconference software so that we could share images of their work and record the session. We centered large portions of the interview on artifacts of the students' work to ground the conversation in the class activity. The researchers then conducted one final interview with Professor X to discuss the work and interview responses of each of the 10 interviewees and his overall assessment of the course.

After all data was collected, the researchers reviewed the student and professor interviews, presentations, homework, and notes files, first to document any insights relative to Research Questions 1 and 2 on Professor X's goals for the course and his enacted strategies to achieve those goals throughout the course. We then made subsequent passes through the data focusing on Research Question 3 for one of the 10 interviewees at a time. We then wrote a narrative detailing each of the 10 student's experience in the course using code words for the 35 emergent categories from the initial analyses when possible. We refined categories and clustered students by similar experience, resulting in 6 subgroups of the 10 students. Rereading the narratives within each cluster, we identified broad characteristics that separated the students into subgroups, resulting in an emergent hypothesis about the effects of student buy-in, goal orientation, and achievement in the course defining their overall experience. We detail the various categorizations with case studies in the results section.

Results

The following articulates our model of Professor X's conceptualization of his own teaching practice relative to the primary goals of *creating proofs* and *overcoming challenges*. The emergent categories from our model appear in italics. They are organized to portray how they all coherently operated in service of Professor X's two primary goals. We then present three accounts of student experience representing variations within the IBL learning environment.

Creating Proofs

For Professor X, creating proof is the heart of mathematical practice and students should not complete a course of study in mathematics without learning how to prove independently. To push students to create proof is to engage them in *real mathematics*. Professor X also believes it gives students *deeper understanding and ownership* over what they learn.

Proof competencies. To successfully apprentice students in creating proofs, Professor X attended to their growth in terms of three requisite skills for proving: *writing/logic*, *ideas*, and *details.* This view of proof writing competence was embedded in his *assessment structure* on homework assignments in which the grades corresponded closely to the proving competencies he wanted to foster. Students could *rewrite* a proof if they earned below a B. Professor X focused the first four weeks of the course on developing writing and logic by allowing students to turn in proofs that other students presented for homework. After that point students could only submit a problem before it was proven in class.

Means of learning to prove. Students were required to *turn in proofs as homework* once per week and *present completed proofs* at intervals throughout the semester. Professor X was readily available for *office hours* and had many students discuss their proofs with him before presenting to the class. In this context he provided *differentiated feedback* targeted at the competence he perceived that the student needed to develop. Students were expected to learn from observing *peer presentations*, which allowed them to see proof approaches that were more accessible to them and help them see proving as an activity in which they could engage.

Reformulated content. To help students independently create proofs in the complex context of Real Analysis, Professor X *reformulated some of the mathematical content*. For instance, the definition of supremum was divided into Right Most Point (RMP) and First Point to Right (FPR), which respectively apply when the supremum is in the set or is not. Neighborhoods were described by order relations rather than absolute value inequalities. Because he valued proof creation, Professor X consistently *ignored elegance and efficiency* in student proving. He generally avoided *demonstrating standard methods*, but rather *legitimized student proofs* so long as they were valid and covered all cases

Differentiated feedback. Many students praised Professor X's feedback. This feedback consisted of *praising and validating student work* and *providing minimal prompts* to move them forward. Professor X articulated gaps in student proofs as lemmas, an instance of *mathematizing student contributions*. Professor X consistently tried to provide minimal feedback so that students retained *ownership* over their created proofs. Providing counterexamples to student proofs was also a way that Professor X *expanded students' concept image* over time.

Creating mathematicians. A final aspect of Professor X's practice related to creating proof was his overarching goal of training new mathematicians. Professor X persistently *invited appropriate students to further study in mathematics*. This invitation had an affective influence, since it represented an expert's high assessment of students' mathematical ability.

Overcoming Challenges

The other overarching aspect of Professor X's view of the value of inquiry-based instruction was pushing students to attempt and complete challenging proof tasks. Stronger students benefitted because they often had never faced challenge in mathematics courses before. Weaker students gained *confidence* by independently writing proofs and presenting them to their peers.

Task difficulty. One key means by which Professor X encouraged students to attempt and overcome challenges was including tasks in his notes that varied markedly in difficulty. Students could not always assess a task's difficulty, leading them to try hard tasks inadvertently. Stronger students who could identify easier tasks turned in proofs for *simple* and *complex tasks* to maintain good grades while spending more time working on *challenge tasks*. Finally, some of the students that Professor X identified (and who self-identified) as mathematically weaker reported attempting problems later in the notes so they had more time to work on them before others presented them in class. Professor X at times withheld *feedback* or guidance from students he perceived as mathematically strong because he wanted to maintain the challenge of the task and their independence in creating a proof. For most students, Professor X very *intentionally praised what was good* in their proof attempts to encourage them to persist in the challenges. Professor X valued how strong students presenting incorrect proofs *legitimized the struggle of creating proofs* on their own, which for him represented *real mathematics*.

The Case of BT

Initially, BT reported being very intimidated by the course, especially because she had to present proofs to her peers. She reported that she started learning more after she could not copy other students' work. Her homework average was a 90, suggesting she earned A's often on her first try. Professor X held a higher view of BT's proving ability than she did. English is not BT's first language, which affected her confidence. Partly due to low confidence, she only presented twice during the semester and thus earned a B presentation grade.

BT reported two significant moments that helped her feel more confident. First, she solved a problem independently and was able to present it to the class, about which she reported:

> For this one, I feel like I figured it out on my own…. So that's why I feel like this was my proud moment proof.… After [this problem] I could definitely see myself growing. That's when I saw that I went from here to here [*raising her hand to indicate levels*]. That's when I started thinking mathematically more…. I felt like I started understanding more, and in a way I started enjoying the class more. I wasn't able to always understand what was put on the board, but I feel like I could grasp the idea, and, in a way, if I ever sat down and worked on it long enough I would be able to prove it.

Later, Professor X personally invited her to take Analysis 2 on the strength of her performance. Regarding Professor X's praise and invitation to take Analysis 2, BT said,

> And I was, "Huh. I might be good in this." I didn't see it, but him telling me this stuff definitely helped me believe it.… Maybe I did get something out of this. Because I don't see it in myself.… He made a big difference…. So in a way I feel like him telling me, "you can do it." It was the push that I needed. Ok, if he sees it then, you've definitely got it. You just need to work on it.

BT was aware of what she did not understand. She looked for problems that made sense to her and often went ahead in the notes to new topics that others had not studied yet. She anticipated this gave her more time and she wanted to make sure the students in the front row who "knew their stuff" would not present it before her. She was also intimidated by indirect proof, even though she felt this led her to write longer case-based proofs. Later in the course, she deliberately attempted proof by contradiction to give herself a challenge. Professor X agreed with BT's assessment of her improvement. She quickly grew to solid B-level work completing basic proofs. She did not achieve "master level" by producing more complex proofs.

The Case of KC

KC was a strong student who expressed appreciation for most all of Professor X's goals and values for IBL instruction. In fact, as a preservice teacher he said he wanted to use IBL in his own future high school classroom because he valued the way it helped him learn. Professor X gave very minimal feedback to KC because he thought he could figure out what was wrong and fix it. KC praised the quality of Professor X's feedback, especially how he could tell him how to correct his proof without letting him know whether the statement was true or false.

KC reported working for long periods on the proofs from this class and enjoying learning. He compared the work in this class to mathematicians' proving, except he felt they had much greater guidance and support from Professor X through the definitions, tasks, and feedback. KC enjoyed challenging problems and often got hooked on trying to see if he could solve them. He had been working extensively on P22 and planned to continue his work after the semester ended if he had not proven it yet. While appreciating the challenge, he said it was humbling to find a problem that he could not yet solve. Throughout the semester, KC was willing to attempt hard problems and Professor X commented that he presented his work well because he had thought hard about the problems at length. Professor X said it was clear that KC 100% liked the course and did hard problems and clear that he had it all in his head, with a "complete and firm grasp of everything."

The Case of SQ

Only one of the 10 interviewees, SQ, was overtly critical of the IBL nature of the course. Others expressed beliefs that implicitly diverged from Professor X's goals. As a preservice teacher, he thought IBL could have uses, but that instruction should usually be more direct.

SQ was very strong mathematically, but expressed frustration over the challenging nature of the course. Professor X told him he would appreciate challenges when he found something he loved and successfully overcame them, but SQ disagreed. He resented the way Professor X pushed him to do more. He wished Professor X guided him toward more efficient approaches.

SQ reported only working on the course homework for the hour before class started, but he performed well due to strong mathematical ability. He thus sought easier tasks and tried to avoid challenges. The variation in task difficulty both allowed SQ to find tasks he could complete easily before class started and allowed Professor X to implicitly push SQ toward more challenging problems if he attempted them after misjudging their difficulty. Professor X offered a telling interaction between the two:

> He probably wasn't doing any more work than looking for the easier problems, then came by my office one time to ask about a question…. I said that's all very good work and very nice and I'm really looking forward to seeing what you do from here, and he said "well I don't really want to work on it anymore." And I said you wouldn't be taking the class if you didn't want a challenge. You didn't come to college because you didn't want challenges. You came to college precisely because you do want challenge. He says, "No. Challenges make me anxious." And he actually was vibrating and sweating. And I noticed that after that when he would ask a question, he would be very nervous. Like it made him very uncomfortable when I would challenge him with a question. And yet I was doing it because he was clearly talented.

SQ seemed to understand Professor X's goals and intentions, even though he did not buy in.

Discussion

Our results are structured as an answer to Research Question 1, identifying Professor X's primary instructional goals of *Creating Proofs* and *Overcoming Challenge*. We organized his classroom strategies to align with these goals in a partial answer to Research Question 2. We now summarize the range of student experiences of the resulting classroom environment illustrated by critical distinctions among our three cases, BT, KC, and SQ, thus answering Research Question 3. We will then return to Research Question 2 and discuss how the various strategies employed by Professor X afforded a wide range of students create their own meaningful proofs and overcome relevant challenges.

Student Buy-in and Goal-Orientation

Our clustering of student experiences and exploration of the range of variation within each group resulted in distinctions along two primary dimensions: students' goal-orientation and their level of buy-in for Professor X's IBL instruction. Dweck & Legget (1988) demonstrated that individuals who viewed intelligence as innate and fixed in an achievement situation typically adopted a goal to demonstrate proficiency, and they persisted only in cases of perceived success while avoiding challenge when they perceive failure. In contrast, individuals who viewed intelligence as malleable and able to grow with use typically adopted a goal to increase their competence, and they persisted seeking challenge regardless of success. BT represents a case of high buy-in to the course goals through a performance orientation. She primarily avoided hard problems and developed pride in being able to complete many of the easier ones for a high grade. This success improved her confidence and she later sought some challenge by branching out to trying proof by contradiction. KC represents high buy-in with a learning orientation. He enjoyed the challenge of the class, even seeking to continue work on difficult problems after the class ended. He appreciated Professor X's IBL approach because it helped him learn and feel like a mathematician, and he wanted to adopt the approach in his own future teaching. SQ represents low buy-in with a performance orientation. Although he and Professor X both assessed that he was more than capable of doing the hardest work in the course notes, he reacted negatively and viscerally to the challenge. He expressed feelings of frustration over both the style of the course and his struggle on problems that he could not immediately solve. We observed no students with learning orientations that did not buy in. We thus generated six categories because we subdivided each of the three categories above between moderate and high achievement in proving.

Differential Engagement of Professor Goals

Professor X consistently expressed goals of developing his students' ability to construct their own proofs and overcome meaningful mathematical challenges. Engaging a variety of students in the class, Professor X adopted several strategies that allowed students to differentially benefit from the course. Based on his judgment of their ability, he sought to give each "a problem worthy of their intellect." He subsequently offered support and feedback to enable them to be successful yet retain intellectual ownership of that success. He challenged students at different levels by offering differentiated feedback withholding (what he judged to be) just the right amount for each student to succeed. He valued success at multiple levels: 1) writing meaningful mathematics and logic, 2) developing key ideas for proofs, and 3) effectively attending to all details for a rigorous argument. He also enabled all students an appropriate entry point by reworking the content to more conceptually accessible units that afford proof without clever techniques, including problems at a wide range of difficulty throughout the course notes. Allowing students to improve and resubmit homework problems supported his focus on overcoming challenges. Professor X continually fostered his students' confidence, initially by allowing them to turn in presented proofs at the beginning of the semester, and always finding some aspect of their work to genuinely praise. Understanding that students would place different value on developing mathematical reasoning, Professor X took the long view of such difference, saying "It doesn't surprise me that many kids have different perspectives, and that's totally ok with me." He simultaneously valued what they got out of it for their current priorities and maintained hope that many would someday come back for graduate study in mathematics.

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