Validating Proofs in Parallel Mathematical and Pedagogical Tasks

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Educators often use tasks that situate teachers in pedagogical contexts, under the assumptions that such tasks activate knowledge authentic to teaching; and, furthermore, purely mathematical contexts may not activate such knowledge. These assumptions are based on analyses that contrast actual engagement with pedagogical context to hypothetical engagement without pedagogical context. We propose that it is important to conduct a direct comparison of responses, and we report on such a study using a set of tasks with and without pedagogical contexts – featuring the same underlying mathematics. The results revealed differences in how secondary teachers validated proof based on context. Context also influenced the importance participants placed on algebraic notation in validating a proof. This study has implications for how and when secondary teachers attend to validity and the role of algebraic notation, and the messages they may convey to their students about validity and notation.

Keywords: Mathematical Knowledge for Teaching, Proof Validation, Secondary Teachers

Many secondary mathematics teachers find their undergraduate mathematical preparation irrelevant to or disconnected from their teaching (Goulding, Hatch, & Rodd, 2003; Ticknor, 2012; Wasserman, Villanueva, Mejía-Ramos, & Weber, 2015; Zazkis & Leikin, 2010). One possible response to this problem is to embed mathematics into pedagogical contexts (e.g., Stylianides & Stylianides, 2010; Wasserman, Fukawa-Connelly, Villanueva, Mejía-Ramos, & Weber, 2016). The strategy behind this design is that situating teachers in *pedagogical tasks*, as opposed to pure *mathematics tasks*, helps activate "the mathematical knowledge needed to carry out the work of teaching mathematics" (Ball, Thames, & Phelps, 2008, p. 395). Because the work on the task resembles work done in teaching, teachers can experience ways in which mathematics applies to teaching, and thus find these experiences useful for their future teaching.

One implicit assumption underlying development efforts is that pedagogical context activates knowledge that is authentic to teaching. Furthermore, this knowledge may not be activated or perceived as relevant in pure mathematics contexts. For instance, all examples of items in Hill, Ball, and Schilling (2008) contain names of students or teachers, and the authors discussed debates of "how much to contextualize [items]" (p. 379) – not whether to contextualize items. Following a description of a task with pedagogical context, Stylianides and Stylianides (2010) concluded, "Presumably, it would be hard for a teacher educator to engage prospective elementary teachers in a discussion of such a subtle but important mathematical issue in the absence of a 'motivating' pedagogical space" (p. 168). Arguments in support of this assumption rely on analyses of teachers engaged in tasks with pedagogical context, contrasted with hypothetical cases where the pedagogical context is absent. Although the arguments are compelling and have advanced the field, there is no empirical evidence for this assumption based on direct comparisons between responses to tasks with and without pedagogical context.

We propose that it is productive to conduct such a comparison. Suppose that tasks set in pedagogical contexts do in fact activate mathematical knowledge differently than mathematical contexts. For example, if different criteria are used to determine whether or not a proof is valid depending on the context, then that may have implications for how mathematical knowledge is used in teaching. Similarly, the explanation of a mathematical idea might have different features when presented with a pedagogical context, which has implications for how that idea would be understood. Differences based on context might reveal unaddressed incoherence in teachers' mathematical knowledge. Differences might also suggest places where connecting undergraduate mathematical content to the work of teaching is particularly difficult.

We hypothesize that if there are differences in responses to tasks set in pedagogical contexts and mathematical contexts, then these differences might be explained by the norms and values teachers hold about mathematics. We base this hypothesis on the observation that different contexts, including orientations toward problem solving, can influence the norms and values brought to bear in solving tasks (Aaron & Herbst, 2012); and that different contexts can prime different knowledge on identical tasks (e.g., Gick & Holyoak, 1980; Ortner & Sieverding, 2008; Yeager & Walton, 2011). In this paper, we report on a study in which we compared 17 high school teachers' responses to parallel mathematics tasks, one situated in a pedagogical context and the other in a university mathematics context. The tasks were exactly the same except for context; see Table 1. We asked: Do teachers validate proofs based on similar norms and values when situated in teaching mathematics compared to when situated in learning mathematics? Our results are highly suggestive that contexts do elicit different orientations to mathematics, in the form of norms and values.

Throughout this paper, we use *pedagogical context* refer to contextual elements of elementary school, middle school, or secondary teaching practice contained in the task text such as student talk or curriculum materials (Phelps & Howell, 2016). In contrast we use *university context* to refer to tasks that are set in the context of an undergraduate mathematics course, and do not have contextual elements related to teaching. Distinguishing these two contexts explicitly highlights the potential differences in teachers' undergraduate mathematical preparation and the mathematical work of their teaching.

Theoretical Perspective and Frameworks Used

Teaching decisions are shaped by orientations (Schoenfeld, 2010), which encompass norms and values. Norms refer to expectations and understandings; values refer to what is perceived as important or beneficial; both have forms specific to the discipline of mathematics (Kitcher, 1984) as well as its learning and teaching (Yackel & Cobb, 1996). The norms and values for mathematics inform those of teaching mathematics, but they are not the same (Ball et al., 2008), and priming with different contexts can potentially activate different resources (e.g., Gick & Holyoak, 1980). Consequently, mathematics teaching entails negotiating mathematical and pedagogical norms and values (Ball & Bass, 2003a, 2003b).

Since formal proof is part of secondary mathematics (NGACBP & CCSSO, 2010), a practice of mathematics learning that arises in teaching is validating mathematical arguments, including proof. The validity and communication of a proof can be contextual (Weber, 2014, 2016). Additionally, Lai and Weber (2014) found that mathematicians would improve proposed proofs differently depending on whether the proof had come from a student or a mathematician.

Data & Method

Rationale

To determine whether the contexts of teaching and learning would elicit different mathematical norms and values, we used parallel tasks. One task featured pedagogical context to situate the participant in teaching secondary mathematics; the other situated the participant as a student in a university mathematics course. We chose to contrast the pedagogical secondary context with a university context because the most recent and intensive context in which teachers experience proofs as learners is university. We determined that these two contexts served as productive contrasts to inform future work in teacher education.

Table 1 shows the set of tasks used to address the first research question. The university context could be considered a pedagogical tertiary context, however, we note that the task situates the participant as a student, not a professor. Moreover, responses from our participants indicate that they were reasoning from the stance of student, not university instructor.

Table 1. Parallel tasks for validating mathematical proofs. The tasks are based on the TEDS-M released item #MFC709 (TEDS-M, 2010).

Pedagogical context	**University mathematics context**
In a unit on mathematical justification, you ask your high school students to prove the following statement:	In a unit on mathematical justification, your mathematics professor asks you to consider proofs of the following statement:
When you multiply 3 consecutive natural numbers, the product is a multiple of 6.	
Below are three responses. Determine whether each student's proof is valid.	Below are three responses. Determine whether each proof is valid.
Kate's answer:	**1.**
A multiple of 6 must have factors of 3 and 2. If you have three consecutive numbers, one will be a multiple of 3. Also, at least one number will be even and all even numbers are multiples of 2. If you multiply the three consecutive numbers together the answer must have at least one factor of 3 and one factor of 2.	
Leon's answer:	**2.**
$1 \times 2 \times 3 = 6$ $2 \times 3 \times 4 = 24 = 6 \times 4$ $4 \times 5 \times 6 = 120 = 6 \times 20$ $6 \times 7 \times 8 = 336 = 6 \times 56$	
Maria's answer:	**3.**
n is any whole number $n\times(n + 1)\times(n + 2) = (n^2 + n)\times(n + 2)$ $= n^3 + n^2 + 2n^2 + 2n$ Cancelling the n's gives $1 + 1 + 2 + 2 = 6$	

Data source

Participants. We interviewed 17 practicing secondary mathematics teachers who had 1 to 14 years of experience teaching, and who had worked with a variety of grade levels and courses.

Tasks. To ensure that the pedagogical context was realistic, we used existing tasks that had been extensively reviewed as representing mathematical knowledge for teaching. For the research question reported, we used tasks, shown in Table 1, based on the TEDS-M released item #MFC709 (TEDS-M, 2010), which represents pedagogical content knowledge (Tatto et al., 2008). The full study considers a second set of parallel tasks, focused on explanation.

Protocol. All participants answered in teaching context first and learning context second, with a distractor between contexts to prime their identity as university students. We asked participants in each context whether they agreed or disagreed with the statement, "Kate's proof/Proof 1 is less valid because it does not use algebraic notation". This question targeted teachers' potential belief in the importance of algebraic notation in proof (e.g., Knuth, 2002).

Analysis. We first coded the reasons why each proof was judged valid or invalid. In the second coding, we looked for differences across context in the determinations about the proofs, their reasoning, and agreement or disagreement about the role of algebraic notation.

Results

Clear differences emerged based on context. Table 2 shows how participants validated each of the proofs, in each context. Table 3 illustrates how participants judged the role of symbolic notation in each context. We now highlight two themes of the teachers' reasoning, and will present the remainder of our results in the full paper.

Table 2. Number of participants determining whether proofs are valid, by context

	Pedagogical context		University context	
	Valid	Not valid	Valid	Not valid
Kate / 1	16	1	5	12
Leon / 2	2	15	0	17
Maria / 3	2	15	3	14

Table 3. Number of participants who disagree or agree: "Kate's proof is less valid because it does not use algebraic notation"/"Proof 1 is less valid because it does not use algebraic notation"

Pedagogical context (Kate's proof)			University context (Proof 1)		
Disagree	Agree	Other*	Disagree	Agree	Other*
14	1	2	3	10	3

*"Other" denotes equivocation, e.g., "If the teacher wants algebraic proof, then yes, less valid. If that's not the learning target, then it's not more or less valid." In the university context, one participant was unintentionally not asked this question, so n = 16 instead of 17.

Privileged norms of communication

In the university context, teachers valued precision and clarity and privileged algebraic notation: "the algebraic notation is clearer, precise, or better than just words, and it is a skill you should have in university." One teacher's explanation of why Proof 1 was not valid, while Kate's proof was valid, captured the privileging of algebraic notation in a particularly remarkable way: "In university, you have to use mathematical reasoning not logical reasoning."

In the pedagogical context, teachers privileged words and focused on explanation: "If you can get down your idea, that's all that matters" or "Using words is important". Some teachers expressed discomfort, wondering whether it was "okay" to hold differences across context. Several teachers insisted that algebraic notation is absolutely needed at the university level, while at the same time not expecting high school students to use algebraic notation.

(Not) Attending to the logical structure of proof

In the pedagogical context, some participants praised Maria's "good start" and stated she needed to explain her work more in order to have a valid proof; these same participants in the university context stated that the reason Proof 3 was not valid was because there was an algebraic error. In both contexts, participants implied that the proof approach would work, for instance saying, "This proof is almost correct, however it is not adequate to simply 'cancel the n's'." One common theme in evaluations of this proof was ascribing validity to the *approach*, even when disagreeing with the details to the extent of calling Maria's proof/Proof 3 not valid. (In fact, the approach would only work with a much more complicated structure that considers cases by divisibility.) There was a sense among some participants that the algebraic approach would eventually lead to a valid proof, especially in the university context.

Significance

Using a novel study design with highly parallel task sets, we contribute a striking example of how the contexts of learning and teaching may activate teachers' norms and values differently. We found that different norms of communication were privileged between the two versions of the tasks. Precision and clarity arose almost entirely only in the university context, and explanation arose almost entirely only in the pedagogical context. The teachers paid explicit attention to algebraic notation, for merits of precision and clarity and because "that's what university professors expect"; teachers at times turned a blind eye to algebraic notation in the pedagogical context, professing that they would be impressed with Kate's work. This contrast raises the issue of how and when secondary students learn to attend to algebraic notation, and what messages teachers send about algebraic notation. Using tasks in varying contexts, especially featuring the same underlying mathematics, can elicit tensions between norms and values about mathematics so that they can be problematized to benefit teachers' use of mathematics in teaching as well as their identities as doers of mathematics.

Questions for the Audience

This preliminary work has helped us shape several questions that we intend to discuss during our RUME presentation. In the presentation, we plan to share a sample of participant work, and discuss how it might change our thinking about approaches to teacher education. We then ask:

1. How compelling is the framing of the problem?
2. We used references from cognitive science to substantiate our hypothesis (that differences in responses to tasks can be explained by differences in norms and values held by teachers in pedagogical and university contexts). Are there results in mathematics education that make an equivalent point or a related point?
3. What are productive strategies for engaging with these data that attend to differences in reasoning across parallel tasks?

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