Partitioning a Proof: An Exploratory Study on Undergraduates Comprehension of Proofs

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In this paper, we explore eleven undergraduate students' comprehension of two proofs taken from an undergraduate abstract algebra course. Our interpretation of what it means to understand a proof is based on a proof comprehension model developed by Mejia-Ramos, et al. (2012). This study in particular examines the extent to which undergraduate students are able to modularize a proof using the proof's key ideas. Additionally, eleven doctoral students in mathematics, referred in this paper as experts, were asked to provide modular structures for the same proofs that the undergraduate students received. We employed experts' modular structures of the proofs to analyze that of undergraduates'. The main finding of the study is that, contrary to experts' proof modularization, undergraduates partitioned the proofs in a way that failed to highlight how key components of the proofs are logically linked, suggesting an inadequate proof comprehension.

Key words: Proof, Proof Comprehension, Modularization, Abstract Algebra.

Mathematics majors are expected to spend ample time on reading and writing proofs. However, despite its importance in undergraduate mathematics education, research on proof comprehension is limited. In fact, much of the proof literature focuses on students' aptitude to construct or validate proofs and less on their ability to comprehend proofs (Mejia- Ramos et al., 2012; Mejia-Ramos & Inglis, 2009). Mejia-Ramos and his colleagues (2009) systematically investigated a sample of 131 studies on proofs and they found that only three studies focused on proof comprehension. They hypothesize that the scarcity of the literature on proof comprehension is perhaps due to the lack of a model on what it means for an undergraduate student to understand a proof. In this study, we used an assessment model for proof comprehension that was developed by Mejia-Ramos, et al. (2012) to explore undergraduates' comprehension of proofs. In particular, this study seeks to examine the to extent to which undergraduates are able to modularize a proof to enhance their proof comprehension.

Theory: Assessment Model for Proof Comprehension

Mejia-Ramos, et al. (2012) proposed that one can assess undergraduates' comprehension of a proof along seven facets. These seven facets are organized into two overarching categories: local and holistic. A local understanding of a proof is an understanding that a student can gain "either by studying a specific statement in the proof or how that statement relates to a small number of other statements within the proof" (p.5). Alternatively, undergraduates can develop a *holistic* comprehension of a proof by attending to the main ideas of the proof. According to the model, students' holistic comprehension of a proof can be assessed by asking students to identify a modular structure of the proof. A good modular structure of the proof shows an understanding of how key components or modules of the proof are logically connected to obtain the desired conclusion.

Review of the Literature

Research looking into students' comprehension of proofs is relatively sparse. In Weber's (2012) study mathematicians reported that they measured their students' understanding of proofs by (1) asking students to construct a proof for a similar theorem to the one that was proven in class, and/or (2) asking them to reproduce a proof; and some said they do not assess their students' understanding of a proof. However, one cannot accurately capture students' comprehension of a proof by having them reproduce it (Conradie & Frith, 2000).

There are fewer studies on what students do when they read proofs for understanding. For example, Inglis and Alcock (2012) conducted a study that compared and contrasted beginning undergraduate students' proof-reading habits to those of research-active mathematicians. By studying their participants' eye movement while reading a proof, they concluded that undergraduate students, compared to the experts in their study, spend more time focusing on the "surface feature" of a mathematical proof. Based on this observation, the researchers suggest that undergraduates spend less time focusing on the logical structure of the argument; this, in turn, seems to explain why students often have difficulty understanding the logical structure of a mathematical argument, as evidenced elsewhere in the literature (Selden & Selden, 2003).

Recent studies on novice proof readers suggests that undergraduates are not successful in gleaning understanding from the proof they see during lecture (Lew et al, 2015). For example, students interviewed in Lew et al.'s (2015) study did not comprehend much of the content the instructor desired to convey, including the method used in the proof. Students interviewed in Selden and Selden's (2003) study also failed to understand a proof holistically since they were fixated on verifying each line and put little emphasis in attending to the overarching methods used in the proof. One purpose of this study is to build on the growing body of research on proof comprehension.

Research Methodology

Participants and Research Procedures

This study took place in a large public university in the northeastern United States. The content of the proof used in this study come from an introductory abstract algebra course. In the chosen research setting the standard textbook used is *Abstract Algebra: An introduction* by Hungerford (2012). The goal of the course (as stated in the syllabus) is to introduce students to the theory of algebraic structures such as rings, fields, and groups in that order.

Since the main purpose of this study is to explore undergraduates' comprehension of proofs—in particular, proofs that appear in an introductory abstract algebra course—the lead author personally approached undergraduates who had taken or were enrolled in an introductory abstract algebra course. Eleven undergraduates agreed to participate in this study and were assigned pseudonyms S1-S11. At the time of the study, six of the eleven undergraduate participants (S3, S5, S6, S7, S8, and S9) were enrolled in an introductory abstract algebra course. Seven participants—S1, S2, S3, S5, S6, S7, S8, and S9—were pursuing a major in secondary mathematics education and said they intended to be high school mathematics teacher. The remaining four students were mathematics majors.

In addition to undergraduates, we used eleven doctoral students, to conduct a finegrained analysis of undergraduates' proof comprehension. At various times, we asked the doctoral students to provide, in writing, modular structures of the proofs. To avoid confusion, in the remainder of this paper we will refer to these doctoral student participants as experts.

In this study undergraduates were given two proofs, proofs A and B found in appendix 1 and 2, and were asked to read for understanding. We chose these proofs for various reasons, including their pedagogical value. For instance, proof A was chosen because it illustrates conditions that one can impose on integral domains to make them fields. Undergraduates were asked to read the proof until they felt they understood it and were encouraged to write and/or highlight on the proof paper as well as to think out loud while reading. Once a participant finished reading a proof, we asked her to (1) partition a proof into its modular structure and (2) explain the purpose of some assertions and how they are logically connected to prove the claim.

Data Analysis

Recall that eleven doctoral students in mathematics were asked to provide a modular structure for both proofs. All eleven modular structures of proof A that the doctoral students provided were studied carefully and resulted in what will hereafter be referred as the expert's modular structure. The expert's modular structure of proof A reads as follows:

> First, fix an arbitrary non-zero element a (lines 1-2 in the integral domain R. Second, using $a \in R$, construct a map from R to R, and then show this map is injective (Lines 1-6). Finally, using results from about maps between finite sets, argue that the map is surjective. It follows then that a has a multiplicative inverse.

Similar to proof A, doctoral students' modular structures for proof B were also studied carefully and the following synthesized expert's modular structure emerged:

> First, if $H = \{e\}$ then the claim follows trivially. Second, consider the case where $H \neq \{e\}$. Using the properties of subgroups and the well-ordering axiom, proof B, in lines 2-4, argues for the existence of the smallest positive integer k satisfying $g^k \in H$. Third, it shows that $< g^k > \subset H$. Finally, using the minimality of k and the division algorithm the proof establishes that $H \subset < g^k >$. It follows then, g^k is the generator of H.

Undergraduates' modular structures of each proof were then analyzed in relation to the expert's using the rubric described in Table 1.

Table 1

Rubric to assess undergraduates' identification of modular structure of proofs

Rating	Criteria
Very poor	• Participant failed to provide a partition of the proof • Participant wrote something completely irrelevant or incorrect. • Participant seems to have copied the claim or a significant part of the proof word for word • Over all, participant described how the proof is structured in way that is *very different* form the expert's modular structure for the proof. This means participant's modular structure of the proof failed to capture the purpose of each module and how they are logically connected.

Poor	• Participant wrote something relevant to the proof, but he/she failed to discuss how each module is related to one another • Participant may have repeated the claim or some ideas or sentences from the proof word for word • Overall, participant described how the proof was structured in a way that has *little resemblance* to the expert's modular structure for the proof
Satisfactory	• Participant partitioned the proof into modules in way that resembles the expert's partition of the proof, but does not always describe the logical relationship between modules • Participant did not state clearly state the purpose of some module or components of the proof • Overall, participant described how the proof was structured in a way that has *some resemblance* to the *expert's*
Good	• Participant explained the purpose of each module and how the modules together prove the theorem • Overall, participant's description of how the proof was structured is *very similar* to that of the expert's

Results

An overwhelming number of undergraduates in this study provided modular structures that suggested that they either *poorly* or *very poorly* understood how the *key ideas* of the proof are logically linked to prove the claim. More specifically, most undergraduates did not identify the purpose of some of the key arguments of the proofs. For example, six students did not correctly address the purpose of showing that the kernel of f_a is trivial (see lines 3-5 of proof A). Table 2 below summarizes our assessment of undergraduates' modular structure of proofs A and B.

Table 2. *Undergraduates' modular structure of proofs A and B*

Evaluation	Undergraduate students	
	Proof A	Proof B
Very poor	S1, S2, S8, S9, S11	S1, S2, S6, S7, S8, S9
Poor	S3, S5, S6, S7	S3, S10, S11
Satisfactory	None	S5
Good	S4, S10	S4

As shown in Table 2, nine out of eleven undergraduate participants provided a structure for proof A that has either little or no resemblance to the expert's. Some undergraduate students, for instance S1 and S2, wrote something that either only amounts to repeating the claim word for word or is a general comment that can be said about any proof, not just proof A. For example, when asked to break the proof into components or modules specifying the logical relationship between each of the modules, S1 wrote: "proof was structured step by step". Another student, S2, said that "[the proof] was structured as a list of consecutive steps…" Note that both S1 and S2 do not provide any thoughts on the modular structure of proof A. Other participants, while correctly describing the goal of the proof A, offered a structure of the proof that is vague. For example, S3 wrote: "The proof started with stating some definitions. Then set some constraints and stated what the goal was. Proved bijection and then the goal which was that every non-zero element has a multiplicative inverse." S3's description of modular structure of proof A's is vague in that many proofs begin with definitions, constraints, and goals. Also, observe that S3 does not mention how the map f_a is defined and the role it plays in proving the claim. S8, on the other hand, attended more to the writing style of the proof and much less about its content. When asked to provide a modular structure of proof A, she wrote:

> The proof had a plan. Step 2 in the proof explains where the proof is going. Step 3 also guides the reader forward letting them know when they are going next. It uses moving language all throughout, words like next, finally, since…

Far from describing key components of the proof and indicating how they are organized to prove the claim, S8 appears to focus on words of the proof rather than the idea of the proof. Also, when asked why in proof A the kernel of the map was shown to be trivial, S8 erroneously stated it was "to support the fact that there exists [sic] no zero divisors." However, the purpose of showing the map was trivial is to show that it is injective. Her response entails that she did not recall one of the properties of integral domain which is the absence of zero divisors. Therefore, it could be the case that her inadequate knowledge of meaning of important terms of the proof such as integral domains might have resulted in insufficient comprehension of the logical structure of the proof.

By contrast, two participants, S4 and S10, offered a structure of proof A that indicated some comprehension of the proof. For instance, S4 structured proof A as follows:

> Lines 1-2 set up that which to be prove
> Lines 3-6 prove that $f_a: R \to R, f_a: x \mapsto ax$ is surjective
> Lines 7-3 prove that f_a is surjective
> Line 9 proves that $\forall a \in R, a \neq 0_R, a$ has a multiplicative inverse

As evidenced above, S4's modular structure of proof A does not have too much detail. Yet, it captures all *key ideas* of the proof that is noted in the expert's modular structure. In particular, both S4 and S10 described the *key* components of proof A; namely, how the map f_a from R to R is constructed and that it is a bijection.

As shown in Table 2, a majority of undergraduates presented a modular structure for proof B that two researchers independently deemed *very different* from the expert's modular structure presented above. For instance, six students, S1, S2, S6-S9, provided a modular structure that is vague and misses *key ideas* of proof B. In describing the modular structure of proof B, S7 wrote: "the proof was divided into components that each proved a 'lemma' that was needed for the next mini proof. All these proofs were needed to prove the claim." Note that S7 does not indicate what the lemmas are and how they were used in the proof. Indeed, what S7 wrote regarding proof B can be said for just about any proof. Moreover, S7 does not correctly identify the purpose of assertions in lines 5-8. She wrote that "…the purpose of these lines [5-8] is to show there does not exist a smaller power of k…"

S4 is the only participant who described a modular structure for proof B in a way that was very similar to the expert's. S4 wrote:

> First, the trivial case (lines 1-2). Next, show that there is a smallest positive integer k such that $g^k \in H$. Finally, prove that $H =< g^k >$ by showing that for all $g^i \in H, i = nk$. (lines 5-9).

First, observe that S4 correctly noted that proof B proceeds by cases. Also, he included key components of the proof such as establishing the minimality of k and using the division algorithm to ultimately show that the any subgroup of a cyclic group is also cyclic. Finally, when S4 was asked to describe the goal of lines 5-8 in proof B, he correctly indicated that the purpose of arguments or statements in lines 5-8 is to show that g^kgenerates *H.*

To summarize, undergraduates in this study demonstrated limited comprehension of proofs A and B. Indeed, six out of eleven provided a modular structure that related *very poorly* with that of the experts, suggesting limited proof comprehension. One plausible explanation for participants' poor proof modular structures has to do with lack of familiarity with the tasks we asked them to do in this study. Stated differently, undergraduates, including those in this study, are rarely asked to partition a proof and asking them to do so might not necessarily reflect their understanding of the proofs. Furthermore, some undergraduates in this study may have viewed these proofs as not long enough to warrant breaking them apart. We suggest that future studies can improve on this study by first showing participants examples on what it means to modularize and then ask students to identify a proof's modular structure.

Appendix 1: Proof A

Direction: Please feel free to write any of your thoughts while reading the proof below. Also please think-out-loud while reading the proof. Note that the numbers only indicate each line in the proof for follow up questions. Below you will find a proof of the following claim.

Claim: Let *R* be a finite integral domain. Then *R* is a field.

Proof. 1. Let *R* be a finite integral domain whose multiplicative identity is 1_R and whose additive identity is 0_R.

2. Since *R* is a commutative ring, it suffices to show that every nonzero element in *R* has a multiplicative inverse.

3. Let *a* be a fixed nonzero element of *R* $(a \neq 0_R)$. Consider the map $f_a: R \to R$ defined by $f_a: x \to ax$. We first show that the kernel of f_a is trivial.

4. Note that kernel of $f_a = \{x \in R: f_a(x) = 0_R\} = \{x \in R: ax = 0_R\}$.

5. Since *R* has no proper zero divisors, $ax = 0_R \Longrightarrow a = 0_R$ or $x = 0_R$. But, $a \neq 0_R$ thus $x = 0_R$.

6. Therefore kernel of $f_a = \{0_R\}$ and so f_a is injective.

7. Next, note that $|R| \geq |f_a(R)|$. Since f_a is injective, it follows that $|R| = |f_a(R)|$.

8. Because $f_a(R) \subseteq R$ and $|R| = |f_a(R)|$, we have that f_a is surjective.

9. Finally, since $1_R \in R$, we have that $\exists x \in R$ such that $f_a(x) = ax = 1_R$. So a has a multiplicative inverse. Therefore, R is a field.

Appendix 2: Proof B

Direction: Please feel free to write any of your thoughts while reading the proof below. Also please think-out-loud while reading the proof. Note that the numbers only indicate each line in the proof for follow up questions. Below you will find a proof of the following claim.

Claim: Any subgroup of a finite cyclic group is cyclic.

Proof. 1. Suppose that $G =< g >$ and $H \leq G$.

2. If $H = \{e\}$, then $H =< e >$. Otherwise, $\exists i \in \mathbb{Z}, i \neq 0$ such that $g^i \in H$.

3. Then, $g^{-i} \in H$. It follows that one of i or $-i$ is a positive integer.

4. The well ordering axiom guarantees that there is a smallest positive integer k such that $g^k \in H$. We will show that $H =< g^k >$.

5. Clearly, $< g^k > \subseteq H$ because H is closed under the operation of G and $g^k \in H$.

6. Suppose that $h = g^i \in H$.

7. By the division algorithm we know that $i = nk + r$ for some $0 \leq r < k$.

8. Then, $r = i - nk$. We have that

$$g^r = g^{i-nk} = g^i g^{-nk} = g^i (g^k)^{-n} \in H.$$

9. Since g^i and $(g^k)^{-n}$ are both elements of H and H is a group, it follows that $r = 0$ or we would have a smaller thank k positive power of g in H. Conclude that $H =< g^k >$

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