Conventions or Constraints? Pre-service and In-service Teachers' Understandings

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Several researchers have noted that it is important for students and teachers to be able to differentiate between what is mathematically critical to a concept or representation and what is a convention maintained for the purposes of communication. In this report, we describe two studies examining the extent to which pre-service and in-service teachers (PSTs and ISTs) understand graphing conventions either as conventions or as rules that must be unquestionably maintained. We highlight the extent to which conventions are pervasive in both PSTs' and ISTs' meanings for graphs and related ideas (i.e., function and rate of change) and describe why such meanings are problematic.

Keywords: Conventions, Graphs, Preservice Teacher Education, In-service Teacher Education

Hewitt (1999, 2001) distinguished between arbitrary and necessary information in mathematics curriculum and learning. He described arbitrary information as that which students need to be informed about by an external source (e.g., the name of an object or representational conventions), whereas necessary information students can deduce for themselves. In addressing graphs and coordinate systems, Hewitt (1999) described aspects of coordinate systems that are necessary (e.g., the need for a starting point or origin and orienting vectors or quantities) and noted:

> These are some aspects of where mathematics lies within the topic of co-ordinates, rather than with the practising of conventions. I am not saying that the acceptance and adoption of conventions is not important within mathematics classrooms, but that it needs to be realised that this is not where mathematics lies. So I am left wondering about the amount of classroom time given over to the arbitrary compared with where the mathematics actually lies. (p. 5)

Whereas mathematicians and mathematics educators are likely to agree with Hewitt's distinction, the extent to which students and teachers maintain understandings consistent with his description is an open question. Hence, in this report, we present the findings from two studies, one with pre-service teachers (PSTs) and one with in-service teachers (ISTs), intended to address the question, "In what ways do pre-service and in-service teachers understand graphing conventions?" In this report, we highlight the extent to which conventions are pervasive in both PSTs' and ISTs' meanings for graphs and related ideas (i.e., function and rate of change).

Theoretical Perspective

Of relevance to this report, when discussing students' use of notation and representational systems with respect to conventions, Thompson (1992) described two ways in which an individual can use a convention: (a) using a convention unthinkingly and possibly unknowingly and (b) using a convention with an awareness that she is conforming to a convention (i.e., convention *qua* convention). Thompson (1992) elaborated, "To understand a convention *qua* convention, one must understand that approaches other than the one adopted could be taken with equal validity. It is this understanding that separates convention from ritual" (pp. 125). We leverage Thompson's distinction in the context of teachers' graphing activity, arguing a teacher's use of graphs entails a *convention qua convention* if the teacher maintains a convention with the *awareness* of maintaining a convention (i.e., understands the convention as one way to represent some idea among other equally valid choices). We claim a teacher's use of graphs entails the *habitual use of "convention"* if the "convention" is a necessary or inherent aspect of a teacher's meanings for graphs and associated topics. In this case, what we as researchers perceive to be a convention is not a convention qua convention with respect to that teacher's meanings; hence, we intentionally use quotations to indicate this difference in perception. As we illustrate in the results section, what an observer understands to be a convention can instead be habitual to a PST's or IST's use of graphs to the extent that the teacher unknowingly assimilates situations in ways that entail the "convention". Alternatively, the teacher might consider using graphs in some different way, but the teacher does not conceive such a way equally valid due to her or his system of meanings necessitating that the "convention" be maintained.

Relevant Literature

Understanding a convention *qua* convention involves an individual being aware of a variety of equally viable representational choices while understanding that a particular choice is customary to a group of individuals. Speaking on various conventions practiced in U.S. and international school mathematics, Zazkis and Mamolo (Mamolo & Zazkis, 2012; Zazkis, 2008) hypothesized that students are hindered in making such distinctions when they only have experiences in which educators maintain particular conventions. Mamolo and Zazkis argued that a potential outcome of educators unquestionably maintaining conventions is that students are not afforded opportunities to develop understandings suitable for novel (e.g., alternative coordinate systems) and unconventional situations.

International and U.S. education researchers (Akkoc & Tall, 2005; Breidenbach et al., 1992; Even, 1993; Montiel, Vidakovic, & Kabael, 2008; Oehrtman, Carlson, & Thompson, 2008) have documented that students often associate function in graphical situations with little more than a ritual application of the vertical line test, a common procedure taught in U.S. school mathematics. As an example, Montiel et al. (2008) identified that students were inclined to apply the vertical line test when investigating relationships in the polar coordinate system. Because some students' meanings entailed carrying out an action tied to the Cartesian coordinate system and a particular axes orientation, those students claimed that relationships such as $r = 2$ do not define a function. In this and other examples (e.g., Breidenbach et al., 1992), the researchers posed graphs that they understood to be representative of functions, yet the students' meanings for functions and their graphs did not afford such understandings.

Our purpose is not to rehash the well-documented claim that students often understand function in unsophisticated ways (see Leinhardt, Zaslavsky, and Stein (1990), Oehrtman et al. (2008), and Thompson and Carlson (2017) for extensive reviews). Rather, our purpose is to draw attention to a particular feature of students' meanings that is more deeply-rooted and problematic than researchers have previously reported. Namely, we infer that the students in these studies drew upon meanings in which what we perceive to be conventions of a particular coordinate system had become features inherent or intrinsic to those students' meanings. For instance, what we perceive to be the convention of representing a function's input along the Cartesian horizontal axis was something the students used habitually (i.e., "convention").

Methods

In order to explore and better understand PSTs' and ISTs' understandings of conventions, we conducted two studies that used similar tasks (see Task Design). In the first study, we designed and conducted 90-120 minute semi-structured clinical interviews (Ginsburg, 1997) with 31 PSTs enrolled at a large state university in the U.S. The PSTs were entering their first semester in a four-semester preparation program for secondary mathematics teachers. Each PST began the program during her or his junior year (in credits), and each PST had completed at least two mathematics courses past Calculus II. We chose participants from the volunteer pool whose schedules aligned with the researchers' schedules.

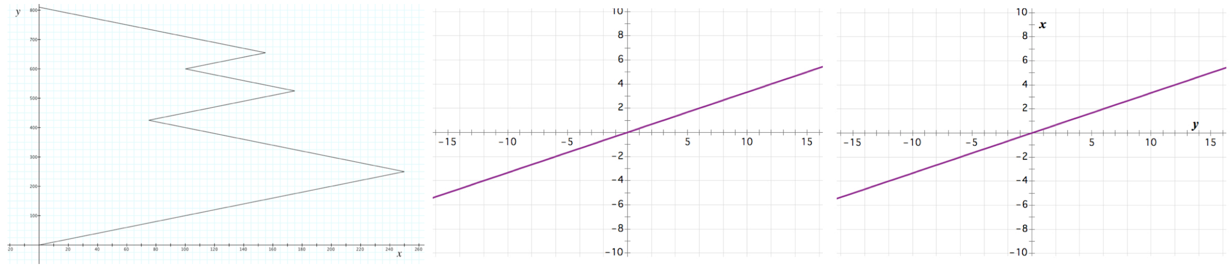
We videotaped the clinical interviews and digitized all written work. We analyzed the data using selective open and axial analysis approaches (Strauss & Corbin, 1998) and conceptual analysis (Thompson, 2008). We identified instances of PST's activity that offered insights into his or her meanings. We used these instances to develop hypothesized models of the student's meanings and we compared a PST's activity across instances and tasks in order to test and improve our interpretations of her or his activity, including identifying themes across instances and tasks. Lastly, we compared across students in order to identify compatible and contrasting meanings. The research team met throughout the data analysis phase in order to refine models of students' meanings and clarify themes in the students' meanings and uses of graphs.

In the follow-up study, we adapted our original tasks for an on-line survey completed by 45 ISTs. The ISTs were geographically distributed across the U.S. and were enrolled in a fully online graduate mathematics course designed specifically for ISTs. We coded the ISTs' responses using open and axial approaches (Strauss & Corbin, 1998) and thematic analysis (Braun & Clarke, 2006). Members of the research team analyzed a subset of the ISTs' responses and we met to discuss our observations, identify commonalities across responses, and adapt or create new codes to capture more ISTs' responses. We iterated this process four times as we refined our codes to capture all ISTs' responses; after obtaining final codes, a second researcher recoded approximately 65% of the data to check for inter-rater reliability. We obtained Cohen Kappa values of 0.78 and 0.85 for the two tasks described, indicating a high level of agreement.

Task design

We designed each task to include what we perceive to be an unconventional feature with respect to the use of graphs in U.S. school mathematics. Because we did not expect the PSTs or ISTs to spontaneously interpret the displayed graphs as entailing unconventional aspects, we designed tasks to include specific claims with respect to features that we intended to be unconventional, often through hypothetical student responses. By including hypothetical responses focused on aspects we considered unconventional, we were able to infer the extent that something was an inherent or habitual aspect of the PSTs' and ISTs' uses of graphs.

To illustrate, we provided the graph in Figure 1a and posed a variant of, "What about a student who claims that this graph represents x is a function of y?" With respect to ***Figure 1***b, we presented the graph as the work of a hypothetical student who graphed the relationship $y = 3x$. We asked the participants to describe how the hypothetical student might have been thinking when creating the graph. The follow-up prompt included a graph with the axes labeled (***Figure 1***c), and we explained that a hypothetical student clarified his graph of $y = 3x$ by labeling the axes as given in the second graph (i.e. x on the vertical axis and y on the horizontal axis). Both tasks illustrate our intent on designing graphs that can be conceived as mathematically viable (albeit unconventional) as presented with respect to the given prompts and claims.



(a) (b) (c)

Figure 1. (a) Is x a function of y? (b) and (c) A hypothetical student's work to graphing $y = 3x$.

Results

We structure the results section by first presenting the PSTs' responses to each task. We then synthesize the ISTs' responses to both tasks. We conclude by comparing the PSTs' and ISTs' responses.

PSTs Responses

Table 1 summarizes the PSTs' responses to the claim, "x is a function of y." Ten of the 25 ('Not true' and 'Unsure') PSTs maintained that the graph does not represent a function due to the graph not passing the vertical line test, because there exist x-values for which there is not a unique associated y-value, or a combination of both. For these 10 PSTs, "function" immediately drew to mind an action that entailed treating (implicitly or explicitly) x or the quantity represented along the horizontal axis as the input quantity (i.e., "convention" as habit). For the seven PSTs who maintained that the statement is true on the condition that the graph is rotated 90-degrees counterclockwise, they understood the phrase "x is a function of y" to necessitate a particular axes orientation—an orientation in which the defined input values are represented horizontally—they required (i.e., habitual use of "convention") that the graph be rotated before considering the validity of the claim with respect to properties of the x-y pairing.

Finally, we interpreted seven of the 25 PSTs' actions to suggest they did not require x or the horizontal axis to represent input values. Yet, five of the seven students hesitated with the claim "x is a function of y" and described that they had a tendency to imagine the graph oriented so that the values defined as the function's input were represented along the horizontal axis. Some students first rotated the graph to determine that the statement is true and then paused when we asked if the statement was also true when considering the graph as given. Ultimately, each of the seven students understood the graph *as given* to be representative of x as a function of y.

Table 1. Codes, counts, and sample PSTs' responses to the statement, "x is a function of y."

Code (value)	#	Sample Responses
True (1)	7	Yeah I guess if you do it this way [*writes '$x(y)$' on paper*]…for every y there is exactly one x. And for every y [*puts marker on vertical axis on graph and moves it horizontally to a point where it hits the curve*] yeah, there's exactly one x…I've never thought about it that way but yeah, he's right…awesome way of thinking about that.
True, if graph is rotated counter-clockwise 90-degrees (2)	8	So she said x is a function of y. That'd be, that'd be looking at it this way [*turning the paper 90-degrees counterclockwise*] and saying look there's no [*motioning hand over the graph as if doing the vertical line test*], there's no crossing…So, I mean that's true, but you'd have to flip the whole graph…[*redraws graph in rotated orientation, labeling the horizontal axis as y and the vertical axis as x*] That'd be y and that'd be x. So x is a function of y. And that's a function…[*Interviewer returns PST's attention to the graph as given*] No, because x isn't a

		function of y. This [*motioning to her sketch*] is the graph of y as a function of x.
Not true or unsure (3)	10	Okay. Um [*pause*] x is a function of y. [*long pause*]…Well you know something's not a function if [*placing her marker in a vertical line over the graph*]… if two different inputs can give you the same output… Which you have here obviously that, you know, these one two three four five six x-values give you different y-values [*using her marker to mark points on the graph in a vertical line*]. I mean these, the same x-value can give you six different y-values.

Table 2 presents a summary of the PSTs' responses to the hypothetical student who graphed $y = 3x$ as shown in Figure 1c. We interpreted each of the 20 PSTs (Table 2, the last two categories) who deemed Figure 1c as incorrect or who expressed uncertainty about the hypothetical solution to hold meanings which entailed the habitual use of "convention." These "conventions" included assigning x-values to the horizontal axis, maintaining particular axes directions for positive and negative values (which arose after rotating the graph), using the horizontal axis as an input quantity (and inferring from the given equation that x represents input values), or a combination of these. Only 11 PSTs maintained that the graph as given in Figure 1c unquestionably represents $y = 3x$. These PSTs identified the graph's departure from convention, and specifically its departure from a customary axes orientation. They also claimed that the departure does not influence the correctness of the represented relationship between x and y.

Table 2. Codes, counts, and sample PSTs' responses to the prompt and graph associated with Figure 1c.

Code (value)	#	Sample Responses
Hypothetical student unquestionably constructed a correct graph (1)	11	He graphed it completely right. That's y equals three x…he's not wrong. He just has a different perspective than the traditional x-y…that's just counter to tradition and normal classroom settings. But I think it's smart of him to understand that it's [the convention] not glued.
Hypothetical student constructed a graph that is both correct and incorrect (2)	11	It's wrong with like how we normally write graphs…So he should lose points because he wrote the graph in like really incorrectly to what, how the graph should be written. Like the horizontal axis should always be x and the vertical axis should always be y. But if you're looking at it based on did he understand that, when y equals three, x equals one, like he understood that, um, relationship between x and y.
Hypothetical student did not construct a correct graph or uncertain if the hypothetical student constructed a correct graph (3)	9	They messed up the placement of x and y…They are looking at it like this right now [*rotating graph 90-degress counterclockwise*]…If you are looking at it this way, it's a negative slope [*tracing graph downward left-to-right*] and it should be a positive slope [*tracing imagined graph upward left-to-right*]…slope is wrong.

ISTs' Responses

For brevity's sake, we do not present the ISTs' responses to each task, as they are compatible with the PSTs' responses. Instead, Table 3 provides the codes we created to capture the ISTs' responses to the hypothetical student work for both tasks, sample responses to the task associated with the graph in Figure 1a, and counts of the number of IST responses coded within each category for each task. We note that only 12 and 25 IST responses for the tasks associated with Figure 1a and Figure 1c respectively indicate they understood conventions qua conventions for these particular tasks. The remaining 33 and 20 ISTs respectively maintained understandings that entailed a habitual use of "convention".

Table 3. Codes description, counts, and sample responses of ISTs pre survey.

Code (value)	Sample Responses to the task in Figure 1a	Fig 1a	Fig 1c
The student's mathematical statement is correct despite breaking from conventions. (1)	That's great! I am so glad you were able to apply the "vertical line test" in a horizontal orientation and realize that you would have a function. You are correct in saying that x is a function of y.	12	25
The student's mathematical statement is true but the student is incorrect because he/she broke from conventions. (2)	I think the student is understanding that x can be a function of y but they are not displaying it correctly through the graph.	9	7
The student's mathematical statement is incorrect or the IST did not address the student's mathematical statement. (3)	It was not a good explanation and x is not a function of y, y is a function of x. The value of y depends on x. They also did not describe what would make it a function.	24	13

Comparing PSTs' and ISTs' Responses

In order to compare the PSTs' and ISTs' responses, we assigned numerical values to each of the three categories within each coding scheme (shown in parentheses in each table). Table 4 provides the average scores for PSTs and ISTs across both tasks. Although these values appear similar, we used a two-tailed Mann-Whitney U-test to examine if there was evidence that the PSTs' and ISTs' responses indicated that they were from different populations. There was not a statistically significant difference between the populations for either task.

Table 4. Average scores of PSTs and ISTs and p-values obtained from a Mann-Whitney U-test.

	Figure 1a	Figure 1c
PSTs	2.12	1.94
ISTs	2.27	1.73
p-value	0.4777	0.2937

Discussion and Concluding Remarks

At the most fundamental level, our findings are significant in that PSTs and ISTs who have completed advanced mathematics courses have developed mathematical understandings that, at best, limit their ability to engage effectively with these topics in situations that we designed to be unconventional. The fact that the ISTs' and PSTs' responses were similar indicates that teaching experience may not have an influence on creating shifts in teachers' meanings with regards to graphing conventions. This finding underscores the importance of giving both populations opportunities to develop more sophisticated meanings for various ideas that are not constrained by, what to us as researchers, are conventional choices. Even more significant is that so many of these teachers (or soon-to-be teachers) held meanings that led to claims and actions that, although often internally viable to them, were contradictory from our perspective and suggested their habitual use of "convention." We do not have the data to comment on the effects of an interaction in which a teacher makes these comments to an actual student who claimed x to be a function of y or who produced an unconventional graph of $y = 3x$, but it is not hard to imagine that the student would be left wondering what he or she did wrong and possibly conclude that axes-variable label pairs and orientations are critical features of a mathematical idea or established rules that must be followed rather than arbitrary conventions (i.e. Hewitt 1999, 2001).

We hypothesize that the PSTs' and ISTs' meanings 'worked' for them throughout their schooling. Due to the pervasive role of conventions in school curricula and instruction, they were likely able to repeatedly assimilate their experiences to these meanings with little or no perturbation. Slope or rate of change associations based on a direction of a line are likely to 'work' in situations that maintain the "conventions" upon which those associations were constructed; function meanings that inherently or tacitly entail the vertical and horizontal axes as representing a function's output and input, respectively, 'work' in situations that maintain those "conventions." Second, due to the PSTs and ISTs repeatedly having the opportunity to construct and re-construct meanings that 'work' without perturbation, they developed a system of meanings that are internally rational and consistent. Such meanings are compatible with what Thompson and Harel (Thompson, Carlson, Byerley, & Hatfield, 2014) called *ways of thinking* – meanings that become so routine or habitual that a person (consciously or subconsciously) anticipates situations involving the associated concept to entail that meaning.

Our intention is not to discredit conventions, nor to convey that conventions are unimportant. Nor do we intend to imply that curricula and educators can realistically be expected to address every convention in mathematics. Instead, we agree with researchers (i.e., Hewitt, 1999, 2001; Thompson, 1992; Zazkis, 2008), who have argued for ensuring that students and teachers become aware of conventions as choices that do not impact the underlying mathematics. One reason for educators to support individuals in becoming explicitly aware of conventions specific to a particular group or field is that conventions vary within and among fields (i.e., it is standard in economics to represent the independent variable on the vertical axis and the dependent variable on the horizontal axis). Hence, collaborating successfully across discipline boundaries requires that an individual become operative with the conventions and practices common to each field, or at least that an individual hold meanings that enable her or him to accommodate conventions of other fields. Another important and more fundamental reason that educators should support individuals in becoming aware of conventions is the restrictions in individuals' ways of thinking that result from constructing a system of mathematical meanings dependent on "conventions". If the goal of mathematics education is to prepare students to provide correct answers in canonical settings, then such restrictions are not an issue. However, if our goal as educators is to support students in constructing a generative mathematics that helps them organize their experiences among and within fields, as well as take on advanced and abstract mathematical ideas that require students to differentiate between what is essential to an idea and what is not, then it is important that they construct mathematical knowledge that has assimilatory capacity in canonical and unconventional settings.

In closing, we argue that our results call into question the entrenched place of conventions in school curricula and instruction. Addressing this issue requires that those designing curricula and instruction take more seriously the negotiation of conventions among students and their teachers. In short, if students and teachers are to understand a convention *qua* convention, then they need opportunities to come to understand mathematical ideas in ways that enable a subsequent negotiation of conventions within the context of those ideas. That is, a productive negotiation of conventions should occur in conversations where a mathematical idea—which is understood as remaining invariant in canonical and unconventional contexts—remains the focus, as opposed to conversations that obscure conventions.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. DRL-1350342 and DRL-1222355. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the NSF.

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