An Undergraduate Mathematics Student's Counterexample Generation Process

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This paper illustrates the processes and struggles involved in a student's generation of a counterexample. The data involves one student's at-home proving while working on homework for his introduction-to-proof course. In this paper, we present an episode where a student engaged in substantive efforts in order to generate a proof by counterexample. We compare and contrast this episode against results from the literature on example generation to provide insights regarding the similarities and differences between example and counterexample generation as they relate to proof.

Key words: counterexample, example space, disproof, undergraduate, at-home proving.

Introduction

Examples and counterexamples are in many ways inextricably linked. Goldenberg and Mason (2008) emphasized this association when they wrote, "In a mathematical context there is little difference between an example and a counterexample: it all depends where your attention is anchored, and what you are attending to" (p. 184). To illustrate this idea, consider $1/\pi$. It is simultaneously an example of an irrational number and a counterexample to the claim that for all real numbers x, $x^2>x$. Given this link between examples and counterexamples, it might be reasonable to extrapolate that the process of generating examples and the process of generating counterexamples are similar. However, since one counterexample can disprove a statement and an infinite collection of examples are insufficient to prove a statement, the underlying reasons for counterexample and example generation in a proof context are fundamentally different. This disparity in purpose may affect the process meaning that in the context of proving, example and counterexample generation processes maybe very different. These two conflicting arguments for why counter-example generation and example generation might be different or similar to each other provide the impetus for the propose study.

Related Literature and Theoretical Perspective

The mathematics education literature has explored the role example generation plays in proving but currently has less to say about the process of counterexample generation and how it relates to proving and example generation. Moreover, the literature provides no definitive answer to how the processes of example generation and counterexample generation relate to each other.

Research on learners' example generation can be broadly partitioned into two categories: studies involving tasks which prompt learners to generate examples and studies involving problem solving tasks which do not explicitly prompt for example generation. Studies with tasks that prompt for example generation often use "and then another" tasks, where students are asked to generate increasingly more examples of a particular mathematical concept (e.g., Watson & Mason, 2002, 2005; Zaslavsky, & Peled, 1996; Zazkis & Leikin, 2007). Typically, these tasks follow a predictable trajectory where first a learner generates immediately accessible examples of the concept (e.g. Goldenberg and Mason, 2008). The learner then works to generate new examples by combining examples and/or varying parameters. The second type of study involves students working on tasks which do not specifically prompt for example generation and highlights instances where example generation occurs as a problem solving strategy. These studies illuminate the utility of example generation as a strategy rather than the process of example generation itself.

Counterexamples

Much of the mathematics education research relevant to counterexamples has focused on their pedagogical uses (e.g., Zazkis & Chernoff, 2008, Zazkis, 1995). Such work, however, does not discuss how students might evaluate a claim to determine that it may be false or subsequently generate counterexamples toward the end of proving a claim to be false. Overall, research on counterexample generation is relatively sparse. Meanwhile, several researchers have worked with graduate students who were prompted to evaluate the truth of mathematical claims (e.g., Alcock & Inglis, 2008, Weber, 2009). Such evaluation typically involves either the generation of a proof that establishes the claim as true or generation of a counterexample showing the claim to be false. However, these studies had either students produce counterexamples in such a quick manner that little can be inferred about the counterexample generation process (Weber, 2009) or focused on documenting the differences in the number of examples/counterexamples generated rather than the process by which they were generated (Alcock and Inglis, 2008). We begin to address this gap in the literature by discussing an episode where a student engaged in substantive efforts in order to generate a proof by counterexample.

Methods

The research study was conducted at a large university in the southwest United States. Data were collected from two introduction-to-proof courses taught by the second author of this paper over two semesters. In this study, we discuss work related to the prompt:

True or False and why: If a and b are both irrational then a^b is irrational.

This task's prompt was intentionally chosen because it does not indicate whether the statement is true or false. As such, the data related to this task includes both students' work to determine whether the claim is true/false and, in the cases where a successful proof was produced, an exploration which led to an appropriate counterexample which forms the basis of a proof. Thus, this task affords the opportunity to not only investigate the process of constructing proof via counterexample, but also investigate the process of how a student may realize the necessity for a counterexample.

Each student in the course was provided with a Livescribe® smart pen and notebook and was instructed to include all work done while working on the assignments, from their initial thoughts on the problems to the final solutions to be submitted for grading. We received 56 assignments with student work relevant to the prompt. The data presented here comes from one of these students.

Results

Given the limited space, this paper's analysis focuses on a single student, Alex, and his associated work on the task. This work involves multiple shifts in notation. It also involves shifts between (1) attempts to formally prove the statement, (2) attempts to disprove the statement via counterexample (counter example generation), and (3) his work on proving related results. The core observation we wish to convey is that these three activities inform each other, with insights and notation from one affecting work in the others. This means that the counterexample generation process described here is more nuanced (and draws from more sources) than the processes of example generation described in the literature. Given the multiple shifts in activities and notation we include Figure 1 below to provide a top level view of these shifts. This also aides the reader in keeping track of how segments of the proving process relate temporally to the process as a whole and when segments of the process are omitted due to space limitations.

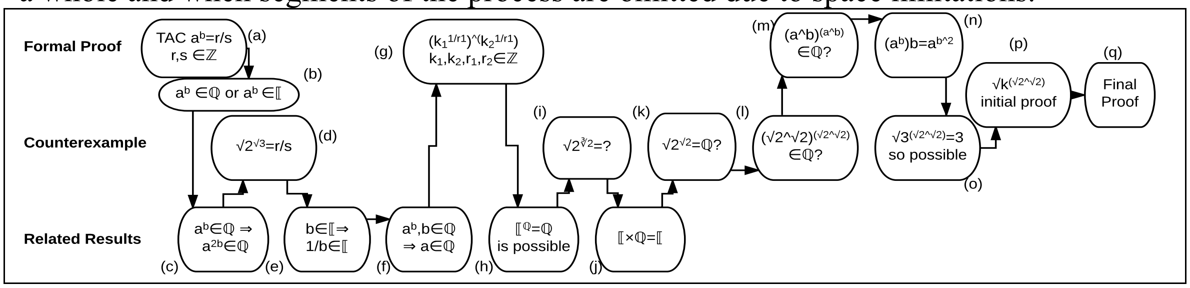


Figure 1. Alex's work on the counterexample task.

Segments a-b: Alex's initial approach to the task

In Alex's initial approach to the problem, he wrote "$a, b \in \mathbb{I} \Rightarrow a^b \in \mathbb{I}$", repeating the statement being considered in the task but using the symbol $\mathbb{I}$ to represent the set of irrational numbers, and "[toward a contradiction] assume $a^b = \frac{r}{s}\ r, s \in \mathbb{Q}$". From here, Alex attempted to syntactically manipulate this equation and arrived at the statement, "$ar^b = s^b$". However, this did not show the contradiction he sought and adjusted his approach to consider both possibilities of the rationality of a^b by explicitly indicating "either $a^b \in \mathbb{I}$ or $a^b \in \mathbb{Q}$". This is the first time where we see Alex considering these two possibilities of rationality. Since he is simultaneously considering both approaches, in figure 1, segment (b) sits between the counterexample and formal proof trajectories.

Segments c-f: Irrational numbers as square roots

After making a note of the set of rational numbers is closed under multiplication, Alex then wrote "Counterexample $a = \sqrt{2}$ [,] $b = \sqrt{3}$[.] Assume $\sqrt{2}^{\sqrt{3}} = \frac{r}{s}$". Here Alex considers the use of specific examples of irrational numbers $\sqrt{2}$ and $\sqrt{3}$, in an attempt to find a counterexample. We note that Alex's choice of irrational numbers is consistent with the typical first examples of irrational numbers in example generation literature (e.g., Goldenberg & Mason, 2008). However, after noting that $\sqrt{2}^{\sqrt{3}}$ is not rational, Alex quickly abandoned this specific counterexample strategy in favor of returning to working with abstract representations to seek a contradiction. We offer Alex's speedy dismissal of his examples $\sqrt{2}$ and $\sqrt{3}$ as evidence that Alex may prefer general counterexamples.

Next, Alex further evoked his knowledge of irrational numbers and exponents, proving the lemma "if b irrational, then $\frac{1}{b}$ irrational". With this lemma established, he attempted to utilize it to create the desired contradiction. In particular, under the assumption that a^b is rational, he wrote, "then $(a^b)^{1/b} = a^1 = a$". This establishes that it is possible to take a^b to an irrational power to yield a. For example, setting $a = 2^{\sqrt{2}}$ and $b = \frac{1}{\sqrt{2}}$, yields a counterexample to the claim. While Alex did not use this approach, we will see that this was a productive stepping-stone toward his eventual solution.

Segments g-j: Irrational numbers as n-th roots

In the next portion of Alex's proof progression, Alex explicitly restricted his consideration of irrational numbers to only roots of natural numbers. This is consistent with his solely considering irrational numbers in his earlier counterexample generation attempts. More specifically, he wrote "if b irrational then b is an r-th root of some number say $b = {k_0}^{1/r_0} \in \mathbb{I}$" and similarly defined the variable $a = {k_1}^{1/r_1} \in \mathbb{I}$ (two lines below, he wrote that $b^{r_0} = k_0 \in \mathbb{N}$.) We note that this example space of irrational numbers is again consistent with the example generation literature (e.g., Goldenberg & Mason, 2008), which identifies n-th roots as the second most accessible class of examples after square roots of non-square integers. Based on this restriction, we cannot say if Alex's example space is restricted to roots or if this represents a restricted evoked example space for the purpose of this task.

Regardless of his understanding of various forms of irrational numbers, Alex uses this new (restricted) representation of irrational numbers to continue his formal, syntactic exploration of a^b as a rational number, where a and b are irrational. In particular, Alex represented a^b as $({k_1}^{1/r_1})^{({k_0}^{1/r_0})}$ and considered the expression $(a^b)^{({k_0}^{1/r_0})}$. He proceeded to again attempt to prove the statement is true via contradiction using this new notation, resulting in a dead end. Beside this work, he wrote several observations related to the original statement. In particular, he acknowledged that a rational number raised to a rational power yields a rational number and that it is possible for an irrational number raised to a rational power to yield a rational number. He justified this latter observation via the specific example $\left(\sqrt{2}\right)^2 = 2 \in \mathbb{Q}$.

Next, Alex wrote "Shows possible $a, b \in \mathbb{I}$ and $a^b \in \mathbb{Q}$ counterexample?" followed by the use of specific numbers ($a = \sqrt{2}$ and $b = \sqrt[3]{2}$) in attempt to generate a counterexample. After noting this did not yield the desired result, Alex returned to his formal exploration using the $({k_1}^{1/r_1})^{({k_0}^{1/r_0})}$ notation. Once again, these attempts were not fruitful and Alex considered the properties of the products of rational and irrational numbers as shown in Figure 2. In this figure, we see Alex once again moving between general and specific attempts to generate a counterexample.

Segments l-m: Numbers of the form $(a^b)^{a^b}$

In the subsequent portion of Alex's proof progression, he expanded his consideration of irrational numbers beyond a's and b's which are roots of integers; he introduced the use of irrational numbers that are composed of a base and an exponent both of which are irrational numbers. This can be viewed as the counterexample generation process being filtered through the task (e.g., Zazkis & Leikin, 2007) where the previously generated pairs of irrational numbers, a and b, are filtered through the task to place a^b under consideration. When a^b is not rational, and thus does not serve as a counterexample to the statement, it can instead serve as a new example of a single irrational number.

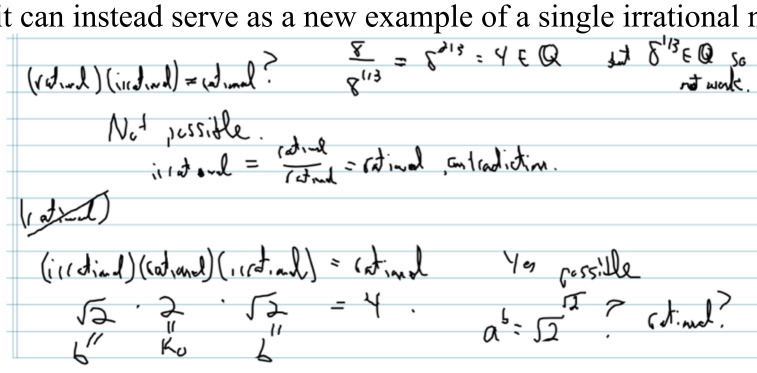


Figure 2. Alex's consideration of the products of irrational and rational numbers.

This behavior first emerged as Alex wrote $(a^b)^{a^b} = a^{b \cdot a^b} = a^r$ indicating the leftmost expression as an irrational base to an irrational exponent and the rightmost expression as a rational number. Alex further considered the possibilities of rationality of the products and powers of irrational and rational numbers. Here $(a^b)^{a^b}$ structure of in his proof attempt is informed by his work with $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}^{\sqrt{2}}}$ when attempting to generate a counter example.

Further, Alex noted that $\sqrt{2}\sqrt{2} = 2$, which led to a reiteration that it is possible to create a rational number by taking an irrational number to a rational power. In this reiteration, he wrote "$a^r = \mathbb{I}^{\mathbb{Q}} \in \mathbb{Q}$ if $a = \sqrt{2}, r = 2$". Given his infrequent use of specific examples and the immediacy of Alex's use of this example following the statement indicating an irrational number raised to a rational power can yield a rational number, we interpret this as Alex reverse engineering a rational number from irrationals and to apply this to the task at hand.

Despite this use of specific examples, Alex next attempted to construct a proof as shown in Figure 3. We see in Figure 3 that Alex's progress was limited when trying to formalize his counterexample. In subsequent attempts to formalize his counterexample, Alex returns to his syntactic representations of irrational and rational numbers rather than attempting to generate a specific counterexample. We offer this as further evidence that Alex may have a preference toward using general counterexamples.

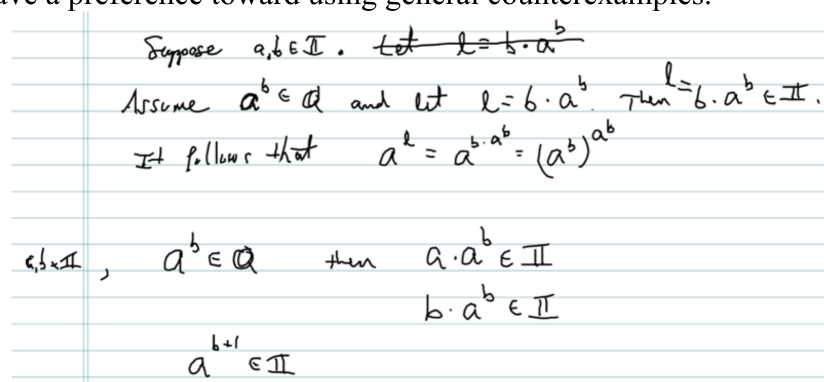


Figure 3. Alex's attempt to formalize a counterexample.

Segments n-q: Numbers of the form $(a^{\sqrt{2}})^{\sqrt{2}}$

Following his unsuccessful attempt to formalize his counterexample in a proof, Alex builds off of his previous approach by modifying the variables and applying specific numbers. Rather than using strictly a's and b's, Alex applied numbers to his previous expression "$a^b = (a^b)^b = a^{b^2} = \sqrt{3}^{\sqrt{2}^2} = \sqrt{3}^2 = 3 \in \mathbb{Q}$" and noted "so possible." This building off of his previous approach is consistent with the literature on example generation (e.g., Goldenberg and Mason, 2008; Zazkis & Leikin, 2007) and is an instance where his attempt to create a proof using abstract a, b notation informed his later generation of a counter example. Further, based on Alex's note of "so possible" next to this expression, we believe Alex has knowingly identified a counterexample that shows the possibility of an irrational base to an irrational exponent to be equal to a rational number. However, it is unclear whether Alex realized that this single counterexample was sufficient to form a basis of a proof that the claim is false. In fact, rather than using the counterexample he generated, Alex continued working towards a general counterexample. One explanation for Alex's preference for general counterexamples may be that his behaviors are an instantiation of the belief that abstract mathematical objects must be the focus of proving activity.

In the proof sketch shown in Figure 4, Alex used $a = \sqrt{k}^{\sqrt{2}}$ and $b = \sqrt{2}$ to show a contradiction that it is not true that if $a, b \in \mathbb{I}$, then $a^b \in \mathbb{I}$. In figure 1, both this and the final proof appear between the formal proof and counterexample axes because they are proofs that utilize counterexamples. It is noteworthy that after having generated a suitable, specific counterexample above ($a = \sqrt{3}^{\sqrt{2}}$ and $b = \sqrt{2}$), Alex generated an additional counterexample ($a = \sqrt{k}^{\sqrt{2}}$ and $b = \sqrt{2}$). Thus, we highlight that the counterexample generation process does not necessarily stop when a counterexample is found. Specifically, in Alex's case, we see that the motivation for this additional counterexample may be focused on supplying a general counterexample. Based on the evidence available, we cannot be sure whether this proof using a general counterexample is a result of 1) a belief that proving activity must usually center around work with abstract mathematical objects, or 2) a lack of understanding of the role a single counterexample plays in relation to a mathematical claim.

If $a, b \in \mathbb{I}$ then $a^b \in \mathbb{I}$.

Either a^b rational or a^b irrational. If $a^b \in \mathbb{Q}$ then disproved.

Assume for any two irrational numbers a, b that a^b irrational

$\sqrt{K}$ irrational if k not a perfect square. $b = \sqrt{2}$ irrational.

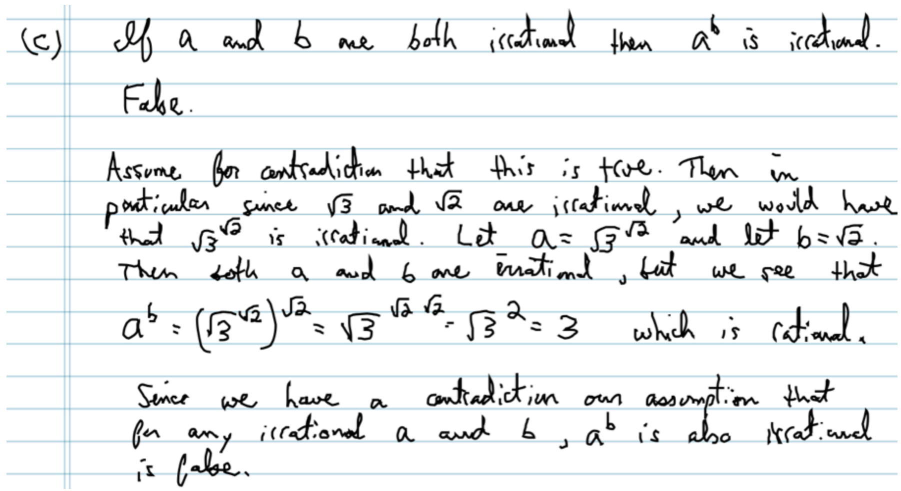
$\sqrt{K}^{\sqrt{2}} \in \mathbb{I}$ by assumption, ~~but if we let~~

But $(\sqrt{K}^{\sqrt{2}})^b = (\sqrt{K}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{K}^2 = K \in \mathbb{Q}$.

Say $K = \sqrt{3} \in \mathbb{I}$

Figure 4. Alex's proof sketch.

Meanwhile, in Alex's final proof, we see that he used the specific counterexample he first presented where $a = \sqrt{3}^{\sqrt{2}}$ and $b = \sqrt{2}$ to show a contradiction that it is not true that if $a, b \in \mathbb{I}$, then $a^b \in \mathbb{I}$. It is unclear what influenced Alex's change from using k=2 to k=3. Moreover, we do not have evidence to indicate why Alex shifted from his more formal approach shown in his proof sketch to using the specific counterexample where $a = \sqrt{3}^{\sqrt{2}}$ and $b = \sqrt{2}$. One possible interpretation of this shift in Alex's approach is the above proof sketch's reliance on the assertion that "$\sqrt{k}$ irrational if k not a perfect square" – a non-trivial claim that warrants a justification.


(c) If a and b are both irrational then a^b is irrational.

False.

Assume for contradiction that this is true. Then in particular since $\sqrt{3}$ and $\sqrt{2}$ are irrational, we would have that $\sqrt{3}^{\sqrt{2}}$ is irrational. Let $a = \sqrt{3}^{\sqrt{2}}$ and let $b = \sqrt{2}$. Then both a and b are irrational, but we see that

$a^b = \left(\sqrt{3}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{3}^{\sqrt{2}\sqrt{2}} = \sqrt{3}^{2} = 3$ which is rational.

Since we have a contradiction our assumption that for any irrational a and b, a^b is also irrational is false.

Figure 5. Alex's final homework submission.

Conclusion

There are several important points to be made about Alex's work on this problem. When he generated examples of pairs of irrational numbers a, b and considered the rationality of a^b, this process was consistent with the example generation literature. This is true not only in terms of his movement from directly accessible examples to less accessible examples, but also the filtering of the example generation through the task itself. However, generating a counterexample was not Alex's primary goal. Rather he was attempting to assess the veracity of the claim and generate a proof that justified that veracity. As a result, the proving process was more complex than generating pairs of irrational numbers a, b in the hope that one had the property that a^b is rational.

The process also included several failed attempts to prove the claim was true via contradiction and the incorporation of several representations of a and b. Moving between these attempts to generate a counterexample (as seen in Figure 1) influenced the generation process in fundamental ways. For example, his use of the pair $a = \sqrt{2}$ and $b = \sqrt[3]{2}$, when attempting to generate a counterexample was influenced by his attempts to use other roots ($b = {k_0}^{1/r_0} \in \mathbb{I}$). Alex's proof attempt encouraged a particular type of example and thus made that notation readily accessible to his counterexample generation process.

Discussion

This proposal offers evidence that proving activities that are not focused specifically on generating examples can influence counterexample generation. That is, proof attempts can influence a students' counterexample generation. The mutual influence of counterexample generation and proof attempts points to the complexity of the process of evaluating the truth of a mathematical claim currently absent in the processes discussed in the example generation literature. This observation regarding the complexity of evaluating the truth of a mathematical claim points to the limitations of the example generation literature for informing this related process. Possible avenues for future research include investigations of learners' potential preference for abstract or general counterexamples to specific ones and more targeted and systematic investigations into the processes by which students generate counterexamples. A natural next step may be to conduct interviews with students working on similar counterexample generation tasks.

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