Physics students' construction of differential length vectors for a spiral path

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As part of an effort to examine student understanding of non-Cartesian coordinate systems and differential elements related to vector calculus, we interviewed students using tasks similar to typical electricity and magnetism problems. In one task, students were asked to calculate the change in electric potential along a spiral path, involving a common line integral. Analysis focused on conceptual understanding and symbolic expression of differential length vectors. Students were heavily drawn to the angular motion of the path through the radial electric field, often only expressing the angular component of the length vector. This contrasts with earlier work, suggesting context may distract from correct mathematical expression.

Key words: Physics, Differential Elements, Vector Calculus, Multivariable

Introduction and Relevant Literature

Students' use of vector calculus in upper-division physics is fundamental to developing an understanding of various principles in electricity and magnetism (E&M). However, the choice of differential elements in vector calculus, specifically line integration, depends on physical symmetries of electric and magnetic fields created by charged objects and current-carrying wires, respectively. As these radial and curling fields are often most simply expressed using non-Cartesian coordinates, the appropriate differential lengths differ from the Cartesian $dx\hat{x} + dy\hat{y} + dz\hat{z}$. The curved paths resulting from angular movement lead to differential length components that are arc lengths and that include scaling factors in cylindrical and spherical coordinates (*i.e.,* $dl_\theta = rd\theta\hat{\theta}$ for spherical coordinates in physics).

Additionally, line integration in physics takes on a different form than in typical mathematics courses. Line integrals having physical application are rarely solved using parametric curves (Dray & Manogue, 2003), and vector calculus in physics is typically non-Cartesian (Dray & Manogue, 1999). The high incidence of symmetry in physical situations allows students to choose a particular component of a differential length vector in a given coordinate system, rather than compute the specific dot product explicitly.

Given the different expressions of differential length elements across coordinate systems and the importance of vector calculus in E&M, we seek to investigate the following questions:

- To what extent do students understand the symbolic expressions and conceptual aspects of differential length vector elements [in non-Cartesian coordinates]?
- How do students use their understanding to construct differential length elements?

Earlier work explored students' understanding of the differential length vector absent of physics context by asking students to construct a length vector in an unconventional spherical coordinate system that we named schmerical coordinates (Schermerhorn & Thompson, 2016a; Schermerhorn & Thompson, 2016b). Current work seeks to build on this by investigating students' solving of line integrals closely related to E&M tasks. This allows insight into how aspects of E&M tasks do or do not influence students' determinations of differential elements, which informs the instruction of differential elements and coordinate systems in physics courses.

Previous work related to mathematics in E&M has sought to address students understanding of integration and differentials where the differential is a scalar element of charge or resistance

(Doughty *et al.*, 2014; Hu & Rebello, 2013; Nguyen & Rebello, 2011). Research attending to student understanding of vector calculus in E&M has primarily addressed student application and understanding of symmetries associated with Gauss's and Ampère's Laws, two common aspects of an E&M course that involve a surface integral and line integral, respectively (Guisasola *et al.*, 2008; Manogue *et al.*, 2006; Pepper *et al.*, 2012). Researchers turning to vector differential operators have explored student understanding and calculation of gradient, divergence, and curl in both mathematics and physics settings, finding student difficulty in interpreting vector fields despite excelling at calculation (Astolfi & Baily, 2014; Bollen *et al.*, 2015; Bollen *et al.*, 2016). Little of this work has specifically explored student understanding of the differential vector element as it appears in the non-Cartesian systems used commonly in physics.

Theoretical Perspective

To explore students' construction of differential length vectors in more typical E&M contexts, we extend theoretical perspectives from previous work on differential length construction in the unconventional coordinate system to allow comparison between tasks. The *symbolic forms* framework (Sherin, 2001) provides insight into students' development of the structure of differential vector elements and determination of how each component is represented in the final equation, while a *concept image* analysis (Tall & Vinner, 1981) gives insight into the particular ideas and aspects to which students attend during construction.

Based on the knowledge-in-pieces model (diSessa, 1993), symbolic forms was developed to explain students construction of expressions when modeling physical situations common to introductory physics (Sherin, 2001). A symbolic form represents the combination of a symbol template and a conceptual schema. The symbol template, an externalized structure such as $\square + \square + \square$, represents the skeleton of an expression containing variables and/or numbers. A student's conceptual schema is the requisite internalized (mathematical) understanding of the role of the template. For example, if students recognized the need to sum multiple quantities that added to a larger whole, they would invoke the $\square + \square + \square$ template. The resulting template-schema pair used here is known as *parts-of-a-whole* (Sherin, 2001).

Meredith and Marrongelle (2008) adapted the conceptual aspects of symbolic forms to describe students being cued to integrate by recognizing reliance on a particular variable (*dependence* symbolic form), or the need to sum up pieces (*parts-of-a-whole)*. The ideas of symbolic forms were expanded to address calculus students' understanding of integrals, often mediated by graphical representations (Jones, 2015). Work exploring physical chemistry students' use of partial derivatives in thermodynamics found that recall mediated students' use of symbolic forms (Becker & Towns, 2012).

A constraint of a strict symbolic forms analysis is that it only yields procedurally based mathematical justifications for the symbolic arrangements and expression structure, neglecting how content understanding plays a role in why the structures or terms are needed. Importing the concept image framework (Tall & Vinner, 1981) from mathematics education rounds out the investigation of conceptual schemata. A student's concept image is a multifaceted understanding including any properties, processes, etc., a student may have about a given topic. A concept image for integration may contain area under the curve or Riemann sums (Doughty *et al.*, 2014). It may also contain a specific rule such as that the indefinite integral of $nx^{n-1}dx$ is $x^n + C$, with or without a specific understanding of why that is the result. By incorporating the concept image framework, the symbolic forms analysis gains a contextual meaning associated with students elicited content understanding, which is not explicitly addressed by the conceptual schemata.

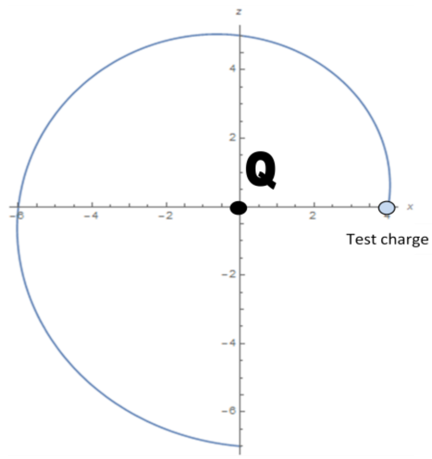

Figure 1: Image given to students, depicting the charges and spiral path of the test charge.

Methodology

In order to investigate students' performance on typical E&M problems students were given a point charge, Q, centered at the origin (Fig. 1). Students were asked to find the differential length vector for a spiral path given by $r = 2\theta/\pi$ in the *xz*-plane and to find the change in potential experienced by a test charge as it moved along the path from the point (4,0,0) to (0,0,-7). The spiral path complicates the task since it requires two differential length components to describe it completely: $d\vec{l} = dr\hat{r} + rd\theta\hat{\theta}$. The electric field due to a point charge is a highly symmetric case where electric potential depends only on changes in the radial direction. For a typical task students only need $dr\hat{r}$ when computing this line integral. This report focuses mainly on students' construction of $d\vec{l}$ to make comparison to generic $d\vec{l}$ construction.

The task was administered in a clinical think-aloud setting with two pairs of students (B&H, D&V) and six individual students (J, K, L, M, N, O) at one university and one individual (T) at a second university. All students were enrolled in the second semester of a two-semester, junior-level E&M sequence. Pseudonyms are provided for students corresponding to their identifying letter (*i.e.,* Jake for J). This particular question took students about 10-20 minutes in interviews.

Video interview data were transcribed and analyzed using a modified grounded theory approach, with the goal of identifying student attention to *symbolic forms* and the associated aspects of students' *concept images* in line with previous findings, while additionally looking for new aspects now appearing because of the applied context. Previously identified symbolic forms include those consistent with Sherin (2001): *parts-of-a-whole*, *coefficient*, and *no dependence*; and new forms to account for the increased mathematical sophistication: *differential* and *magnitude-direction* (Schermerhorn & Thompson, 2016b). The concept images often spurring the need for these templates or necessary terms included *component and direction*, *dimensionality*, *differential*, and *projection*, as well as specific associated actions, such as *recall*, *grouping*, and *transliteration* to other coordinate systems (Schermerhorn & Thompson, 2016a).

Results

Data analysis showed attention to many of the relevant symbolic forms and concept images identified in the schmerical differential length task, but among fewer students.

In particular, *parts-of-a-whole* (PW) and *magnitude-direction* (MD), both prominent in the acontextual task, were generally absent for students' construction in the spiral task. Five students invoked PW, described earlier as students' recognition of parts summing up to a whole with the template □+□+□. However, only one applied a polar coordinate system and initially included MD. MD accounts for the magnitude and unit vector parts of a quantity and is associated with the template □$\hat{□}$. Both these symbolic forms are associated with the *component and direction* concept image, where students would recognize that differential length vectors need multiple components, and that each component corresponds to motion in a specific direction. The following transcript illustrates a correct response and highlights the *component and direction* aspect needed for differential length vector construction:

> Molly: Yeah, and then you go a little bit…I'm picturing you go from this point to this point …So first I travel in the r direction so I go dr in the $\hat{r}$, and then I travel in the $\hat{\theta}$ direction and the arc length of a circle is the radius times the angle that you move so that is $rd\theta$, here in the $\hat{\theta}$. (Fig. 2a)

Molly appropriately separates each component as two distinct motions ("I travel"), then encodes each length as the magnitude and the corresponding direction as the unit vector, resulting in a correct $d\vec{l}$.

Two other students invoked PW without encoding components with a MD template. Neither student specifically attended to the directions each component traced out, resulting in differential length components absent of unit vectors (Figure 2b, 2c). Kyle's transcript demonstrates this:

> Kyle: We stay in the one plane… so we're only changing by θ and r, so it we have some $d\theta$ or let's say $\Delta\theta$, then Δr is going to be $2\Delta\theta/\pi$, so the actual length is the change in the radius and the change in the angle times the radius so that we stay in units of length.

Upon recognizing a need to account for a dot product during the later integration, both students added unit vectors to each of their terms.

Both of the above transcripts also highlight students' multiple concept images of the *differential*, accounting for "a little bit" of or "changes" in variables, consistent with students' ideas of differentials identified in the literature (Artigue *et al.*, 1990; Hu & Rebello, 2013; Roundy *et al.*, 2015; Von Korff & Rebello, 2012). These ideas cue students' invocation of the differential symbolic form: representing a differential quantity with template d□.

The last two students to invoke the PW template used Cartesian coordinates. They both mentioned needing small changes in x and y, rather than starting in a more appropriate polar coordinate system. Oliver attempted to differentiate coordinate transformations for x and y with respect to θ in order to express dx and dy. Tyler began similarly but then suggested that a spherical transformation would produce $dl = r^2 \sin\theta\, drd\theta$. He reduces his dl down to one component without addressing a need to maintain a sum of two components, or directionality.

The remaining interview subjects only attend to one component, neglecting both the PW and MD symbolic forms. Dan and Victor addressed just the change in the r direction, ignoring the change in θ as irrelevant to calculation (Fig. 3a). While this does lead to the correct solution for the potential difference, the length element for the path is incomplete without the θ component.

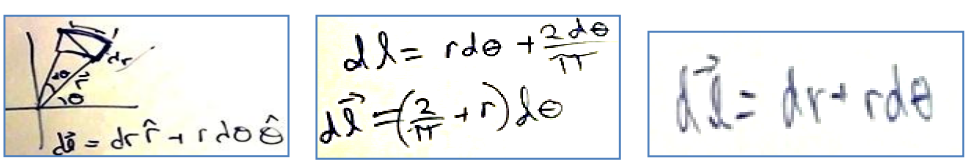

Figure 2. Left to right: (a) Molly's correct differential length elements. (b) Kyle's and (c) Jake's differential length elements absent of unit vectors.

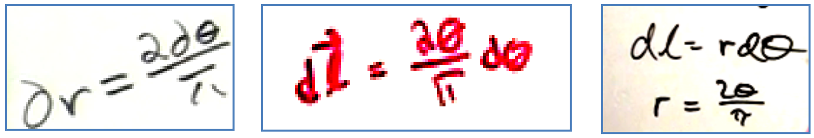

Figure 3. Left to right: (a) Dan and Victor's accounting for only change in r-direction and converting to terms of θ. (b) Nate's dl, with function replacing r in $rd\theta$. (c) Bart and Harold's dl, where the function for r is written with the term to account for changes in r along the path.

The three remaining students only account for the θ component (Figs. 3b, 3c), correctly including the r in the arc length and including the functional relationship to write the length component in terms of θ:

> Nate: I think I'm going to move just a tiny bit. This point changes, and so r is going to change and [θ] is going to change… r is going to be obvious because I think it's going to be [$2\theta/\pi$] and then [θ] would just change some d[θ]… To me it makes sense, because you're moving some infinitesimal amount in θ and then you have that r change.

This reasoning appeared across multiple interviews. Students still recognize the need for change in particular variables, an evoked concept image that results in the differential symbolic form. Here students use the functionality of r on θ and the inclusion of r in arc length to account for r changing. This appears to supersede their need to include change in r as a separate component of the differential length. The need to include a dr is entirely absent from their constructions.

Conclusions

Analysis of student interviews on differential length construction on a more typical E&M task reveal that students are not as attentive to the vector nature of differential elements compared to similar construction in the unconventional spherical coordinates. This may be due to familiarity with the high symmetry of many tasks in E&M that allow students to select one component of a length or area vector and disregard others. Typically for a task involving a spherically symmetric electric field, students would usually select the $\hat{r}$ component. However, students interviewed on the spiral task are largely only selecting the θ component. For these students the change in r is activated, but where in schmerical coordinates this would result in an expression of dr, it appears the salience of θ changing and a functionally dependent r being a variable in the arc length, allows students a justification for their choice of one component.

Student use of θ was prominent. Almost all students, even those expressing multiple components, worked to express their final differential length vector in terms of θ, despite the simplicity of integrating a radially dependent field in terms of r. This focus on θ is most likely due to the salience of the circular nature of the path and/or the functional form of r given to students. It is additionally possible the recent familiarity with circular symmetry in E&M II and Ampère's Law, played a role in students' emphasis on the θ component.

Whereas students easily recognize the need for multiple components for the general expression of the differential length vector, in this more typical task embedded in a physics context, students have difficulty recognizing the need to separate out directions. We seek to investigate students' work on these tasks without a function for the path, to see if this leads to inclusion of the $\hat{r}$ term. Current instructional implications speak to more emphasis on connecting whole differential length vector construction to the determination of terms based on symmetry arguments However, more work is needed to make specific claims regarding students' choices.

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