Connecting the Study of Advanced Mathematics to the Teaching of Secondary Mathematics: Relating Arcsine to the Study of Continuity, Injectivity, Invertability, and Monotonicity

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Prospective mathematics teachers are usually required to complete courses in advanced mathematics to be certified to teach secondary mathematics. However, most teachers do not find these advanced mathematics courses as relevant to their teaching. In this paper, we describe a novel way to teach real analysis to future teachers that connects the content of real analysis to the activity of teaching secondary mathematics. We illustrate this method by describing a module that links the study of the relationship of continuity, injectivity, and strict monotonicity in real analysis to the teaching about the arcsine function and solving trigonometric equations in secondary mathematics. We describe a teaching experiment in which this module was implemented and present evidence of the efficacy of this instruction.

Keywords: Inverse; Real analysis; Teacher preparation; Trigonometry

In the United States and elsewhere, prospective secondary mathematics teachers are required to complete extensive coursework in undergraduate mathematics to become certified to teach secondary mathematics. This coursework usually includes advanced upper-level coursework for mathematics majors (e.g., CBMS, 2001), with many institutions currently requiring that future mathematics teachers complete the equivalent of an undergraduate degree in mathematics (Ferrini-Mundy & Findell, 2001). However, many teachers find the advanced mathematics courses that they complete as irrelevant to their teaching (e.g., Wasserman et al., 2015; Goulding, Hatch, & Rodd, 2010; Rhoads, 2014; Zazkis & Leikin, 2010). In this paper, we focus on how we can design advanced mathematics courses to better meet the needs of prospective teachers.

Relevant literature

The influence of advanced mathematics on subsequent teaching

Although prospective secondary mathematics teachers are usually required to complete many courses in advanced mathematics, several scholars have noted there is little research on whether or how these courses influence prospective teachers' future pedagogical practice (e.g., Deng, 2008; Moriera & David, 2007; Ticknor, 2012). Here, we discuss two findings that suggest that completing such courses have only a modest effect on prospective teachers' pedagogical behavior. First, large-scale studies have found a weak relationship between the number of advanced mathematics courses that a teacher has completed and the achievement of that teacher's students (Darling-Hammond, 2000; Monk, 1994).

Second, when practicing secondary mathematics teachers have been asked how their experiences in advanced mathematics courses have influenced their teaching, many teachers claimed that their advanced coursework did not contribute to their development as teachers (e.g. Goulding, Hatch, & Rodd, 2000; Rhoads, 2014 Ticknor, 2012; Zazkis & Leikin, 2010). For instance, Zazkis and Leikin (2010) surveyed or interviewed 52 practicing secondary mathematics teachers about how their understanding of advanced mathematics influenced their teaching. The majority of the participants in Zazkis and Leikin's study claimed that they rarely used their knowledge of advanced mathematics in their teaching and few could cite any specific instances of their knowledge of advanced mathematics actually informing their teaching. Wasserman et al. (2015) found that this occurred even when the teachers demonstrated an understanding of the advanced mathematics that they were taught.

Reasons why advanced mathematics may not benefit prospective mathematics teachers

Researchers have proposed two reasons for why advanced mathematics courses might not benefit prospective mathematics teachers, even if the prospective teachers understood the content that they were studying. The first reason relates to what Klein (1932) has referred to as a "double discontinuity" between K-12 mathematics and advanced mathematics: the K-12 mathematics that students learn bears little resemblance to the advanced mathematics that is taught at universities and the advanced mathematics that prospective K-12 mathematics teachers learn in university is irrelevant to their future pedagogical practice. In the last decade, researchers have explored this double discontinuity in more detail.

A primary reason that advanced mathematics can inform the teaching of secondary mathematics is because there is an overlap between the content covered in advanced mathematics and the content and disciplinary practices covered in secondary mathematics (Wasserman & Weber, in press). For instance, a first real analysis course deals with concepts such as the real numbers, functions, continuity, and inverse functions, all of which are important concepts in high school algebra, trigonometry, pre-calculus, and calculus. Even though the same concepts and disciplinary practices are covered in advanced mathematics courses and secondary mathematics courses, the way these concepts and practices are treated differs significantly. For instance, Moriera and David (2007) presented a theoretical analysis of how advanced mathematics courses framed concepts from the secondary curriculum. Moriera and David noted that in advanced mathematics courses, concepts usually were introduced using a single canonical formal representation. For example, the familiar concept of fractions was defined as an equivalence class of ordered pairs in $\mathbf{Z} \times \mathbf{Z}\backslash\{0\}$ where (a, b) and (c, d) were equivalent if $ad = bc$. However, Moreira and David argued that effective teaching of secondary mathematics often required the use of multiple representations, many of which were visual but not necessarily formal. For example, fractions might be represented both numerically and pictorially as pie charts, which students will not usually witness in an advanced mathematics course. Similarly, continuity is defined in advanced mathematics formally in terms of epsilon-delta definitions. This treatment bears little resemblance to the informal graphical manner in which continuity is treated in secondary mathematics (e.g., Tall, 2012; Winslow, 2013). Consequently, teachers who study concepts such as fractions and continuity in advanced mathematics may see few implications for teaching secondary mathematics because the advanced treatment of these concepts will not meet the needs of their students (Deng, 2008).

A second disconnect between the activities that in which university students engage in advanced mathematics and in which instructors engage while teaching secondary mathematics (e.g., Ticknor, 2012). For instance, students in advanced mathematics spend a substantial amount of time studying and producing proofs. However these proofs would be usually be inappropriate to use in secondary mathematics classrooms because they employ technical vocabulary, abstraction, and methods of reasoning beyond what secondary students are capable of following (Wasserman et al., 2015). It is not obvious how studying and writing proofs should inform pedagogical activities such as designing activities, grading students' work, and providing informal explanations that a secondary student can understand. Further, prospective mathematics teachers often think these links are non-existent (Wasserman et al., 2015).

Theoretical perspective

Why prospective mathematics teachers must take advanced mathematics: A trickle down model

From our perspective, the anticipated benefits of having prospective teachers complete a course in advanced mathematics can be modeled by the "trickle down" model presented in Figure 1 that considers the relationships between i) advanced mathematics; ii) secondary mathematics; and iii) *teaching* secondary mathematics. This model highlights that most of the material covered in an advanced mathematics course consists of advanced mathematics, where little attention is paid to secondary mathematics. However, the hope is that the advanced mathematics provides an opportunity for the prospective teacher to better understand certain aspects of the content of secondary mathematics. For instance, by learning the zero divisor property about rings in abstract algebra, the prospective teacher may develop a deeper understanding for why you can solve polynomial equations by factoring polynomials (e.g., Murray & Star, 2013). Or by engaging in disciplinary practices such as proving, the prospective teacher may develop a better appreciation about the nature of those disciplinary practices (e.g., Even, 2011). Some instructors of advanced mathematics may be explicit about the connections between advanced mathematics and the content of secondary mathematics, but in often prospective teachers are asked to make the connections themselves. Next, the expectation is that prospective teacher's better understanding of the secondary mathematics content will inform their future *teaching* of mathematics. In our experience, exactly how prospective teachers should teach differently is rarely discussed in advanced mathematics courses. Prospective teachers are expected to use their understanding of advanced and secondary mathematics to improve their teaching on their own or the connections between advanced mathematics and teaching secondary mathematics will be provided in a subsequent education course (Murray & Star, 2013).

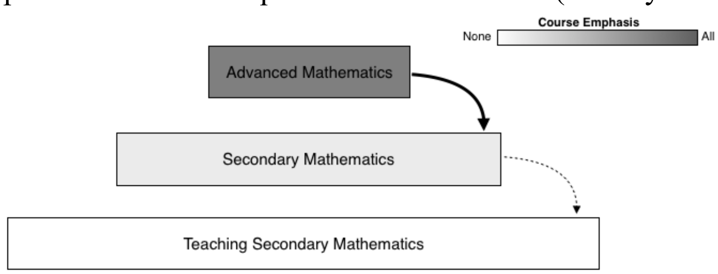

Figure 1. Implicit model for real analysis courses designed for teachers

Our alternative model for teaching advanced mathematics to prospective teachers

We present an alternative instructional model for how advanced mathematics can be taught to prospective teachers in Figure 2. We begin by presenting a realistic pedagogical situation from secondary mathematics. From there, we discuss the secondary mathematical concepts that are in play and problematize the mathematical challenges inherent in the situation that we provide, highlighting fundamental issues that lie beneath the surface that are handled in real analysis. Next we discuss the issue in terms of real analysis. We cover the associated concepts with a formal treatment and make explicit what connections this has for high school mathematics. Finally, and importantly, we describe how this knowledge can inform our response to the initial pedagogical situation that we posed in the beginning of the lesson.

We designed our pedagogical situations to satisfy three criteria. First, there should be a relationship between the real analysis being taught and a topic from the Core Curriculum State Standards in Mathematics (CCSSM, 2012). We do this so that the topic is present in the secondary mathematics curricula and we are not merely preparing students to engage in enrichment activities. Second, the pedagogical situation invites or requires students to engage in what Deborah Ball and her colleagues refer to as a "High Leverage Practice" (HLP), where an HLP "is an action or task central to teaching" (TeachingWorks, 2013). HLPs include providing explanations or models to explain a concept and analyzing and critiquing instruction for the purposes of improving it. We used HLPs so the teachers were engaging in activities that are central to their practice. Finally, we strove to create situations that PSTs would perceive as authentic.

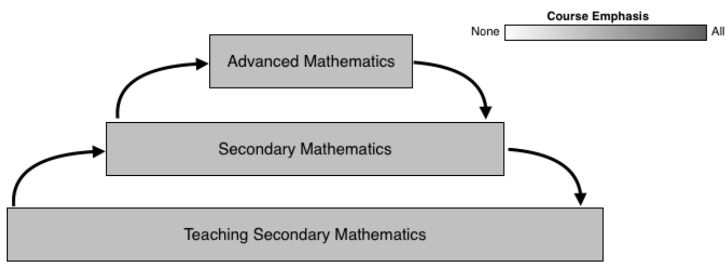

Figure 2. Our model for real analysis courses designed for teachers

Research methods

Broad research context

The data reported in this paper are part of a larger study supported by the National Science Foundation. Our analysis focuses on the 7th of our 12 modules. The real analysis covered in this module include definitions and theorems concerning the relationship between continuity, injectivity, and strict monotonicity. Particularly important is the theorem that a continuous function is invertible on an interval if and only if the function is strictly monotonic on that interval. The secondary mathematics that we cover involves introducing the arcsine function to a trigonometry class and grading and providing feedback on a student's incorrect solution to a trigonometric equation. Hereafter we refer to this model as the Trigonometry Module.

In this paper, we report on the third iteration of a teaching experiment in which the Trigonometry Module was implemented. The Trigonometry Module was initially informed by a study in which we probed 14 prospective and in-service teachers understanding of inverse and the arcsine function as well as the relevance of real analysis for understanding these topics (Wasserman et al., 2105). We developed and implemented the Trigonometry Module, first in an unpublished constructivist teaching experiment (Steffe & Thompson, 2000) with three PSTs and then in the context of a university real analysis course with 32 prospective and in-service teachers (Wasserman, Weber, & McGuffey, 2017). Following the principles of design research (e.g,. Cobb et al., 2003), we used our analysis of the first two iterations of our Trigonometry Module to refine our models of PST's thinking, how we anticipated the PSTs would engage in our activities, and the activities themselves. For the sake of brevity, we do not describe these refinements here, but will do so during our talk.

Data collected from this iteration

The third iteration of the Trigonometry Module occurred at a large state university in the northeast United States. At this university, mathematics education undergraduates were required to complete a mathematics major and a real analysis course was a requirement for this major. In spring 2017, the mathematics department offered x sections of real analysis, one section of which was advertised as a special section that was taught by our research team; the third author of the paper was the lead instructor of the course. This section was advertised as a special section of the course for prospective teachers, although the course was open for all mathematics majors. In total, 17 students enrolled in the course, 13 of these students were in the mathematics education program, two expressed an interest in teaching, and two did not express an interest in teaching.

The class met three times a week. Roughly one out of the three weekly class meetings was devoted to implementing a real analysis module, one of the three weekly class meetings consisted of a traditional lecture covering real analysis content that we did not find relevant to teaching secondary mathematics (e.g., compactness and uniform continuity), and one of the three weekly class meetings was a workshop in which students were given practice and assistance solving problems and writing proofs. When we implemented our modules, we had the students sit in four groups of three to five students. For three of the groups, a member of the research team observed and facilitated the group's discussion. We collected the following data: We videorecorded all of the instructor's actions, we audiotaped each group as they worked on the activities in the module, we had electronic copies of the students' reflective journal entries, performance on a pre-test and post-test, and their homework, and we archived the instructor and researchers' field notes.

Analysis

Following the principles of design research (Cobb et al., 2003), prior to conducting our instruction, we had anticipated models for how the PSTs would understand the central concepts of our module, such as inverse function and the arcsine function, desired understandings that we wanted students to develop by the end of the module, and a hypothetical learning trajectory (Simon, 1995) for how students' engagement with the module's activities would foster these desired understanding. For each activity in the module, we had anticipated behaviors for how the PSTs would engage with each activity. These theoretical models were informed by experiences teaching trigonometry (Weber, 2005), prior laboratory studies in which we interviewed PSTs about their understanding of inverse functions and the arcsine function (Wasserman et al., 2015), and significantly by the first two iterations in which we implemented the Trigonometry Module. In the retrospective analysis (Cobb et al., 2003), we analyzed the extent that PSTs' actual behavior aligned with our anticipated behaviors and developed theories to account for any revisions.

In our analysis of the pre-test and post-test data, we first coded PSTs' responses for mathematical correctness. One item asked students how they would introduce the arcsine function to their class. We noted whether PSTs said in their explanations that the arcsine function was the inverse of the sine function with the domain of $[-\pi/2, \pi/2]$ (a correct response) or that the arcsine function was the inverse function of sine with no domain restrictions (an incorrect response). Another item presented PSTs with the a situation in which a student presented the following solution to a trigonometric equation:

$$\begin{aligned} \sin(2x) &= .7 \\ \arcsin(\sin(2x) &= \arcsin(.7) \\ 2x &= 0.7754 \end{aligned}$$

x = 0.3877 + 2πk

This solution contains two mistakes. The correct solution is x = 0.3877 + πk and 1.1831 + πk. So the student work contained two errors. When taking the arcsine of both sides, the student neglected one of the solutions in the [-π,π] interval (π – arcsin(.7) = 2.3662) and the student added the 2pk at the last step, rather than the third, thus missing solutions of the form 0.3877 + πk when k is an odd integer. In evaluating the PSTs response to the student's work, we documented which mistakes, if any, the PST identified. After coding for the mathematical accuracy of the PSTs' responses, we analyzed the ways in which the PSTs would respond to students qualitatively and interpretatively using thematic analysis (Braun & Clark, 2006).

Results

Lesson and behaviors

Given space constraints, we give only a quick synopsis of what transpired in our implementation of the Trigonometry Module, but a more extensive analysis will be presented in our talk. As we anticipated from our prior research (Wasserman et al., 2015), PSTs did poorly on the pre-test. Most thought that sin *x* and arcsin *x* were inverse functions and few spotted either of the mistakes in the student solution that they were asked to evaluate.

To step up to secondary mathematics, the instructor provided the students with the definitions of injective functions and (strictly) monotonic functions. Then PSTs were asked to explore the relationship between continuity, strict monotonicity, and invertability by debating about whether four statements were always true, sometimes true, or never true. For instance, one statement was that if a function was strictly monotonic on an interval, then it had an inverse on that interval. (This is always true). The purpose of these activities was to engage PSTs in "productive struggle" as they wrestled with these ideas so that the subsequent real analysis would be motivated. Next, we stepped up to real analysis by having the lecturer present two theorems. The first theorem was that strictly monotonic functions were always invertible. The second theorem was "Let $I \subseteq \mathbb{R}$ such that I is an interval. Suppose f(x) is a continuous function from A to $\mathbb{R}$. Then f(x) has an inverse if and only if f(x) is strictly monotonic."

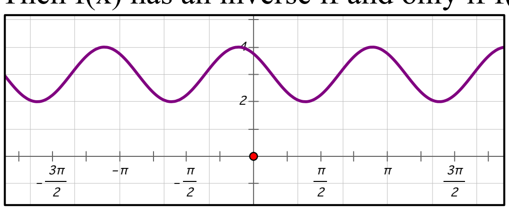

Figure 3. Graph of *f(x)* that students were asked to consider

We had the PSTs step back down to secondary mathematics. PSTs were given a worksheet specifying that "If we want an inverse for a continuous real-valued function *f(x)* but *f(x)* is not one-to-one, **by convention,** we seek to find the largest interval *A* on which *f(x)* is monotonic such that *A* contains 0 and at least one positive number". They were then asked to revisit the relationship between sin x and arcsin x. They were also shown a graph of *f(x)* in Figure 3 and asked to identify the conventional domain restriction in which *f(x)* would have an inverse, if *f(x)* would have an inverse on domains such as [π/4, π/2], and how they could justify their answers using the theorems that were previously discussed.

The lecturer presented a proof of the following theorem: Suppose *f(x)* is a continuous real-valued function *f(x)* that is one-to-one on an interval *I*. Suppose $a \in f(I)$. Then, $x = f^{1}(a)$ is unique solution to the equation $f(x)=a$ in the interval I. The lecturer discussed how in secondary mathematics, periodicity and symmetries were used to find the solutions to equations of the form $f(x) = a$ outside of the domain in which $f(x)$ was conventionally restricted. Finally, in stepping back down to practice, in the post-test, participants were again asked how they would introduce sine to the students and then they were asked to provide feedback to students who solved the trigonometric equation cos(3x) = .5, making errors similar to the students on the pre-test.

Pre-test and post-test comparison

On the pre-test, when asked how they would introduce arcsine to students, only one of the 10 PSTs mentioned domain restrictions. Eight PSTs said they would begin their presentation explaining the nature of inverse and that arcsin x was the inverse of sin x with no mention of domain restrictions. On the post-test, nine of the ten PSTs explicitly mentioned domain restrictions (with the remaining PSTE vaguely saying that he would explain that "arcsin x undoes the sine function to a certain extent"). Many PSTs used creative student-centered activities to illustrate the points (e.g., presenting students with the graph of sin x on the blackboard and inviting students to erase parts of the graph until the function had an inverse).

On the pre-test, only one PST found both errors in the student-generated solution, one PST found one error, one PST gave an ambiguous response, and the other seven PSTs found no errors, focusing instead on issues such as the student "rounding too early". On the post-test, seven PSTs found both errors, two PSTs found one error, and one PST gave the same ambiguous response that he did on the pre-test. The PSTs generally provided feedback to the student by pointing out how arcsin x was only the inverse of the sine function on its restricted domain.

Summary and Discussion

In this proposal, we presented evidence that by studying real analysis while participating in our Trigonometry Module, PSTs were better able to engage in high leverage practices about teaching inverse trigonometric functions. This includes providing an explanation or a model for explaining what the arcsine function is as well as evaluating and providing feedback to a students' argument. We have provided evidence that PSTs lacked the mathematical knowledge to engage in these HLPs effectively before our module; they gave mathematically incorrect explanations about the meaning of arcsin x and they did not recognize mistakes that a student made in his solution to a trigonometric equation. After the class, most PSTs did not make these errors. In the talk, we will also document how they provided pedagogical responses that were not only mathematically correct, but thoughtful and appropriate.

More broadly, we have described a pressing issue—PSTs are required to take advanced mathematics courses but are not benefitting from doing so. We have described an innovative method for addressing this problem by linking the content of real analysis to the high leverage practices that PSTs must engage in. Finally, we have provided an illustration of a module built in accordance with our theory and refined from several iterations of design research, along with evidence that we have achieved our desired learning goals when we implemented this module. Consequently, what we are presenting is a theoretically driven existence proof that our innovative model has the potential to make advanced mathematics relevant for practicing teachers.

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