An APOS Study on Undergraduates' Understanding of Direct Variation: Mental Constructions and the Influence of Computer Programming

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This study explores undergraduates' understanding of direct variation before and after instruction using computer programming to teach generalization over the concept. Based on an initial genetic decomposition for direct variation, the four math/CS researchers developed a research design that included lessons featuring computer programming and mathematical proof writing activities. This study shares results from an application of the instructional research design to N=33 undergraduates interested in teaching. Lessons were from a secondary education math methods course. Follow up interviews were conducted with seven participants. The analysis, using APOS as a framework, categorized mathematical behaviors at the Action, Process or Object level. The study identified obstacles that may have prevented progression through deeper levels of understanding such as deficient prerequisite skills and an affinity for routine algebraic manipulation rather than considering underlying relationships. The student data demonstrated how computer programming activities influenced undergraduates' mental images.

Key words: Direct Variation, Generalization, Computer Programming, Pre-Service Teachers

Introduction

The ability to generalize is considered an essential skill for reasoning about and deeply understanding mathematical concepts by mathematics education researchers. Many researchers have investigated how to explicitly induce students to develop generalizations in the context of mathematical explorations (Tall et al., 1991). Many mathematics education researchers believe that using computer programming activities designed to parallel the construction of an underlying mathematical process may stimulate or accelerate the development of the associated mathematical construction (Dubinsky and Tall, 2002; Authors, 2012). In prior work, we developed an explicit method for motivating students to generalize into mathematical constructions using computer programming exercises and proof writing based on the theoretical perspective of APOS theory. In this study, we extend our previously published preliminary report examining student's understanding of direct variation (Authors et al., 2016). The research questions we investigate are: (1) Does our genetic decomposition of direct variation adequately describe the observed students' constructions; and (2) Do our instructional treatment's computer programming activities influence students' mental constructions as described in the genetic decomposition?

Proportional Reasoning and Theoretical Framework

Mathematics education researchers have dedicated considerable energy to proportional reasoning with elementary and middle school students, high school students, undergraduates and graduates. Collectively these researchers have shown that students and adults have difficulty with problems involving proportional reasoning (Noelting 1980; Vergnaud, 1983; Hart, 1988; Lesh et al., 1983; Kaput and West, 1994). We found that explanations for the difficulties undergraduates experience with the concept of direct variation are sparse in existing literature. Hence this study will contribute to the literature on how undergraduates understand direct variation by examining students' mental constructions and exploring how computer programming activities support the development of mathematical constructs.

The theory of reflective abstraction was described by Piaget (1985) as a two-step process, beginning with reflection on one's existing knowledge, followed by a projection of one's existing knowledge onto a higher plane of thought. Further, Piaget (1985) and Dubinsky et al. (2005a, 2005b) wrote that during the process of cognitive development, reflective abstraction could lead to the construction of logico-mathematical structures by the learner. The conviction that reflective abstraction could serve as a powerful tool to describe the mental structures of a mathematical concept led Dubinsky to develop APOS theory.

In APOS theory the mental structures are Action, Process, Object, and Schema. A mathematical concept develops as one acts to transform existing physical or mental objects. Actions are interiorized as Processes and Processes are encapsulated to mental Objects. It is tempting to view the progression as linear, but APOS practitioners hold that learners move back and forth between levels and hold positions between and partially on levels. In other words, the progression is not linear. This nonlinear behavior and the resulting mental structures may explain the different ways learners respond to a mathematical situation (Arnon et al., 2014).

Genetic Decomposition for Direct Variation

The genetic decomposition was developed as a conjecture of the mental constructions, Actions, Processes, and Objects, that may describe the construction of mental schema for the concept of direct variation as it develops in the mind of the learner. The genetic decomposition served as a model for the design of this research study as well as the analysis of the results. It was also the basis for the computer activities in the lessons that were developed for the students. The pervasive impact of the genetic decomposition is consistent with an APOS theoretical framework (Asiala, et al., 1996).

The prerequisite concepts to start the construction of direct variation are an Object conception of multiples of a number, a Process conception of variable and an Object conception of constant. The notion of equality (=) needs to be used as a relation between corresponding elements of two sets. The learner must have a Process level conception of one-to-one correspondence between two sets X and Y, and be able to recognize and compare corresponding members.

Action

The Actions needed are simple algebraic manipulations involving division and/or multiplication of numbers. The learner will apply the Actions to substitute in known values and solve for an unknown value in the equation. For example, she or he might find a constant (k) by dividing the first value (x) by the second value (y), and then multiply a subsequent number by k to find the answer. Each activity is viewed by the learner as a single instance, isolated from subsequent similar instances. At this level, k is viewed as a specific value, not as an arbitrary constant. The learner may or may not see the relationship between x and y, they may work several examples without seeing a general pattern.

The same Actions described above can take place in different settings with different representations of the relation, such as a table, mapping, graph, and an analytical example.

Process

These Actions are interiorized into Processes as the learner repeats the Action with different values of k or different values of x or y. They might iterate through values of x, but instead of checking specific numbers, the student can determine in general and in his or her imagination, for example, that as values of x increase, corresponding values of y will increase. The learner recognizes a general behavior that x and y vary.

As the learner iterates over x, this Process with x, y, and k is coordinated into a new Process where the learner can view a sequence of numbers X and can determine if elements x in a set X vary with corresponding values y in a set Y without multiplying each value of x by k but by imagining each value of x as a multiple of its corresponding value of y. While they imagine multiplying by k or dividing by x and y to get k, they may not see that y is locked into a value by x and k, into a pattern that is carried out no matter what value is given. They may or may not see the rate of variation as a constant rate.

Object

The Process of checking if elements of a sequence of numbers X are equal to a constant multiple k of corresponding values of Y, (or quotient of x and y is constant) encapsulates into an Object when the individual is able to apply Actions or Processes to it. The Actions that can be carried out on the Process conception of direct variation include comparing and contrasting it with other generalized properties of multiples such as doubling or halving, and to interpret the role of varies directly in the possibility that the two sets X and Y have a constant k when any corresponding elements are divided. For example, they may understand that the ratio between corresponding elements of X and Y is a constant k. They may double the values in X and observe that values in Y are doubled. Then they may halve the values in X and observe that values in Y are halved, and so on. The learner may generalize the process that the subsequent values are determined by k, the constant of proportionality, locked in a pattern that is carried out no matter what value of x you select. Another Action on the process may be reversing the process to determine X when k and Y are known.

Methodology

We applied our instructional treatment to the concept of direct variation for this study. Our investigation was carried out with 33 upper level undergraduates who were interested in teaching mathematics. Each subject participated in a complete lesson including the pre-test, response sheets, and post-test. The format of the lesson was as follows. A brief introduction to the programming environment was given along with the code template shown in Figure 1. A cursory review of the relationship distance is rate times time ($d=rt$) was also presented. Using the code template with an increasing rate and fixed time, participants were asked to complete the program to output the associated distance. Learners were encouraged to experiment with their computer programs and make observations about any relationships. Once this initial table was constructed, the participants were ushered through a series of program modifications and written responses. For example, they were asked to add columns to their programs to depict the doubling or halving of the rate with time fixed and the resulting distance. Programs were modified to show the results of doubling, tripling, and halving the rate with time fixed. Written responses to questions and reflections on their observations were recorded by the participants on their response sheets including generalizations of behavior. Observations on variation and direct variation were solicited as general expressions and participants were taught how to denote the general expressions in mathematical language. For example, participants might observe that if rate doubles and time is fixed, then distance doubles. The instructional treatment was designed so that repetition with various program modifications would stimulate the desire to generalize the observed behavior and make conjectures about the mathematical construct. The final stage of the lesson involved making conjectures and convincing arguments. Participants were shown how to use general expressions to support, or refute, a conjecture using mathematical language. They were then asked to attempt their own convincing arguments with the general expressions they recorded during their inquiry. All of the participant's responses were collected on written response sheets during the lesson. Additional data was collected in the form of interviews. We recorded interviews with seven of the participants which were then transcribed and analyzed. All of the collected data was reviewed and scored using APOS theory. We devised a ranked set of scores to denote pre-action, action, process, and object levels for the direct variation concept based on our genetic decomposition and recorded scores for each subject's pre-test, response sheets, post-test, and where applicable interview data. In the event that authors disagreed, a discussion and further analysis of the data was used to reach consensus.

```
print("r", "t", "d", sep="\t")
t = 5
r = 1
while r < 11:
  print(r, t, "?", sep="\t")
  r = r + 1
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Figure 1. Computer programming template for lesson

Results

In the discussion that follows, R denotes the researchers and U0001 to U0033 identify undergraduates. Results are presented that show how student mathematical behavior correlated to the genetic decomposition. Results also illustrate the influence of computer programming on students' ability to generalize over the concept of direct variation.

Action

Student responses to questions were scored at Action level based on the description in the genetic decomposition. Action level responses were analyzed by the authors for common mathematical behaviors. Student responses during the lesson and in interviews following the lesson fell into three categories of mathematical behavior:

- Category 1. Using specific values or thinking about specific instance
- Category 2. Balancing the equation
- Category 3. Substituting a value in the equation

Action Category 1. Using Specific Values or thinking about specific instance. In the follow-up interview, the researcher asked the student to explain their thinking on a response.

R: What were your thoughts on this? (pointing to post-test response)

U0001: I like having values just cause[sic] it helps distinguish what we're already going over like variables are fine but when I actually have a number to place with the variable it makes it easier to keep up with where I'm going and what I'm doing. So I would place a random value somewhere just so I know how to get from point A to point B.

Algebraic manipulations of a general expression. It is not unexpected that students at the Action level for a concept would use specific values to direct their problem solving. Surprisingly, this study found that ten of the eleven Action level students did not rely on specific values but performed algebraic manipulations on a general formula. What looked like a general argument, which might imply an Object conception, was instead an explicit, step-by-step procedure to balance the equation. This is similar to Frith, et al. (2016) who found students could work proportion problems applying “mechanical knowledge or algorithmic procedures” without actually reasoning about the relationship. Mechanics of algebra included either trying to balance the equation (9 instances) or to substitute general expressions into the equation (8 instances). Students at this level did not meet the prerequisite skills, as defined in the genetic decomposition, two students were at the pre-action level for the concept of multiples, eight did not meet the prerequisite process level for the concept of variable, two did not meet the prerequisite for constant, and one did not meet the prerequisite for the concept of one-to-one correspondence.

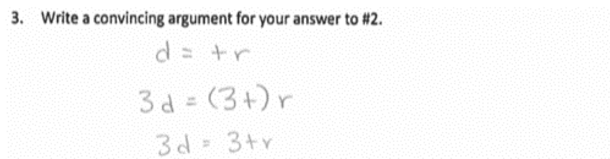


Figure 2. Action Category 2 – Balancing the equation

The snip of student U0005 in Figure 2 shows a typical response in Action Category (2). The student carried out the step by step procedure, multiplying both sides of the equation by a constant, e.g., if $d=tr$ then $3d=(3t)r$. This student wrote in their response of a “need to balance”, as they multiplied both sides by 3.

The snip of student U0029 in Figure 3 shows a typical response in Action Category (3). The student carried out the step-by-step procedure, substituting $3r$ for r in the equation $d=rt$. This work demonstrates a lack of the prerequisite requirement for a process understanding of variable, as d takes on the role of the first distance and the second distance.

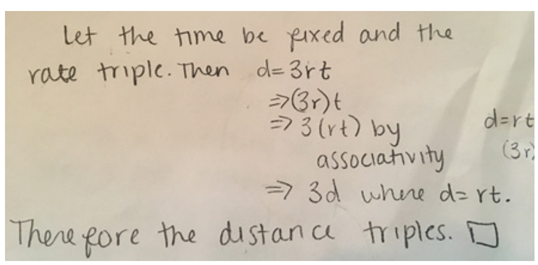


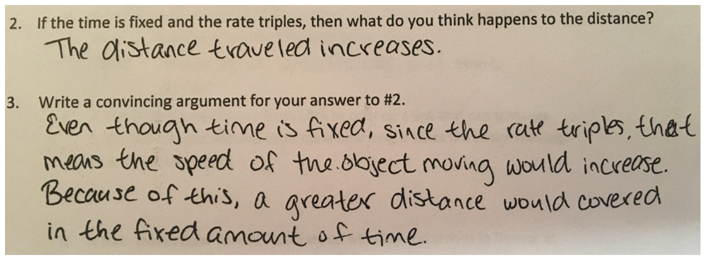
Figure 3. Action Category 3 – Substitution

Process

Of the 14 students at the process level, 10 demonstrated the notion of varies without demonstrating a notion of varies directly. Student concept of varies fell in two categories: (1) x increased (or decreased) then y increased (or decreased) or (2) x increased (or decreased) by some multiple, then y increased (or decreased). In either case, whether or not they repeated the given information about x, for their part in the solution they did not mention the multiple. They did not indicate an awareness of the “locked relationship” between x and y that is determined by the constant of proportionality k.

A typical response for varies in Process Category (1) was demonstrated by student U0003 who described a dependence between rate and distance where the rate increased then the corresponding distance “will increase as well”. The parenthetical statement by the student “The same time frame in a quick pace" indicated they were imagining a process in their mind, where rate and distance varied in a coordinated way.

The snip of work from U0007 in Figure 4 shows a typical response for varies in Process Category (2). The student described a dependence between rate and distance where the rate tripled then the corresponding distance traveled increased. They are imagining a process where an object is moving at a faster speed so "a greater distance would be covered in a fixed amount of time". There was a Process in their mind where rate and distance varied in a coordinated way.


2. If the time is fixed and the rate triples, then what do you think happens to the distance?
The distance traveled increases.

3. Write a convincing argument for your answer to #2.
Even though time is fixed, since the rate triples, that means the speed of the object moving would increase. Because of this, a greater distance would covered in the fixed amount of time.

Figure 4. Process Category (2) - Varies

In neither case did the students in Process Category (2) demonstrate a knowledge of the "locked in" relationship between x and y that is a part of direct variation and is fully determined by the constant of proportionality.

Object

The responses that indicated Object level understanding of direct variation, according to our genetic decomposition, fell in two categories: (1) Relationship between X and Y locked in place by k, (2) Elements of X were dependent on values in Y and the dependency determined by k. Although 32 of the 33 students correctly identified two general expressions relating d, r, and t, only two students demonstrated an Object level knowledge of the fixed relationship between X and Y, determined by k, before the instructional treatment.

Influence of Computer Programming on Generalization

The influence of writing computer programs to explore the concept of direct variation was demonstrated by 16 of the 33 students. These students referenced their programming activities in their responses, in multiple instances, even though neither the question (nor the instructor) suggested responding with program code. Students naturally and intuitively adopted language from their programs. Twelve of the sixteen, who referenced programming in their responses concerning general expressions, improved at least one level during the instructional treatment, while two stayed the same and two went down a level. The students who referenced their programs when asked to give a general expression fell into two categories: (1) Computer Input: Print Statements and (2) Computer Output. In both cases illustrated below by typical responses, the students imagined generating code in their mind, and copied their imagined code onto their response sheet.

The response from Student U0007 in Figure 5 shows a typical response for Computer Category (1). The student imagined writing a computer program with the displayed print statement as an input statement. The response below was after the first computer programming activity. The print statement was stuck in between the answers for Response #3 and Response #4. It appears as a transition between the English statement "the distance is also doubled" and the general expression *(2r)t*. The transitory work is seen as the student wrote "*d = r2*t*" above the print statement "*(r*2)*5*".

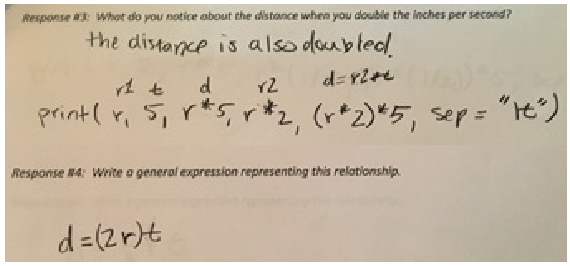


Figure 5. Category (1) – Computer program

Two students demonstrated evidence that running a computer program in their mind and reflecting on the output in table form was a transition from English language to mathematical language. The response from Student U0002 demonstrated the typical response for Computer Category (2) by constructing a table for Response #5 in the left margin after concluding, "halving the rate also halves the distance". The same student then responded in Response #6 with "d2 = $\frac{rt}{2}$.".

In the following interview snip, U0004 described how they developed a "mindset of generalizing" during the instruction. They did not demonstrate a general notion using variables in their expression until after the first programming activity.

R: Just describe when you were writing the last couple of proofs or either one of the proofs
U004: I was thinking more of just the letters and generalizing it after we had done those together and the ones on the other response sheets because I think I was in a mindset of generalizing it…
R: Right
S: So the way I wrote it out, I put more notation the second time on the post-test.

Conclusion

In this study, students explored direct variation through an explicit method for teaching generalization that uses computer programming and convincing arguments. The researchers found that scoring and assessing undergraduates' conception of direct variation was complex due to the task-dependent and context-dependent nature of conception. The genetic decomposition adequately described the students' constructions observed in the data. We noticed students at the Action level tended to manipulate algebraic expressions without understanding the underlying structure. We found many students in our study have a notion of vary but not directly varies. We observed some students who needed to construct the property vary, at the Process level, before constructing the property varies directly, at the Object level and conjectured that a notion of vary is a prerequisite to directly vary. Therefore, we have modified our genetic decomposition to account for this in future studies. We found that prerequisite deficiencies corresponded with the inability to progress through levels of understanding as measured by APOS. We found that students naturally turned to their computer programs to help find general expressions for the concept. Some students considered the inputs to their programs and others reflected on the outputs of their programs when asked to write general expressions for observed relationships. The programming activities influenced students and served as a catalyst to move from purely English descriptions of their conceptions to using mathematical symbols and a "mindset of generalizing".

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