Student Understanding of Linear Combinations of Eigenvectors

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Student understanding of eigenspace seems to be a particularly understudied aspect of research on eigentheory. To further detail student understanding of eigenspace relationships, we present preliminary results regarding students' reasoning on problems involving linear combinations of eigenvectors in which the resultant vector is or is not an eigenvector of the matrix. We detail three preliminary themes gleaned from our analysis: (a) using the phrase "is a linear combination of" to support both correct and incorrect answers; (b) conflating scalars in a linear combination with eigenvalues, and (c) reasoning about the dimension of eigenspaces versus a number of eigenvectors.

Keywords: Linear Algebra, Student Reasoning, Eigenspace, Linear Combination

Purpose and Background

Linear algebra is particularly useful to science, technology, engineering and mathematics (STEM) fields and has received increased attention by undergraduate mathematics education researchers in the past few decades (Dorier, 2000; Artigue, Batanero, & Kent, 2007; Rasmussen & Wawro, in press). A useful group of concepts in linear algebra is eigentheory, or the study of eigenvectors, eigenvalues, eigenspaces, and other related concepts. Eigentheory is important for many applications in STEM, such as studying Markov chains and modeling quantum mechanical systems. Despite this importance, research specifically focused on the teaching and learning of eigentheory is a fairly recent endeavor and is far from exhausted.

One aspect of eigentheory that seems to be particularly understudied is eigenspace, including how students understand linear combinations of eigenvectors. Some research on eigentheory has included eigenspaces but not as the main focus. For instance, Salgado and Trigueros (2015) found that students struggled to construct the concept of eigenspace as well as to coordinate the number of eigenvectors corresponding to a given eigenvalue with the dimension of the space spanned by the eigenvectors of that eigenvalue. Gol Tabaghi and Sinclair (2013), on the other hand, found that exploration of a two-dimensional "eigen-sketch" in Geometer's Sketchpad helped students understand the existence of multiple eigenvectors for a single eigenvalue as they dragged the vector $\boldsymbol{x}$ along the line of the eigenspace. Lastly, Beltrán-Meneu, Murillo-Arcila, and Albarracín (2016) gave students a test question asking if various linear combinations of eigenvectors in $\mathbb{R}^2$ would also be eigenvectors; they found students either reasoned symbolically by explicitly verifying the eigen-equations for the numerically given matrix and vectors, or formally by reasoning about the resultant vectors belonging or not belonging to an eigenspace.

In order to more explicitly explore students' understanding of eigenspaces and extend research beyond 2×2 matrices, the research question for this study is: How do students make sense of and reason about linear combinations of eigenvectors?

Theory and Literature Review

This report is part of our ongoing effort to analyze students' understanding of eigentheory. In doing so, we ground our work in the Emergent Perspective (Cobb & Yackel, 1996), which is based on the assumption that mathematical development is a process of active individual construction and mathematical enculturation. In this report we focus on the mathematical conceptions that individual students bring to bear in their mathematical work (Rasmussen, Wawro, & Zandieh, 2015). The literature on the teaching and learning of eigenvectors and eigenvalues points to several aspects of eigentheory that are important as students build their understanding. Here we summarize that literature by highlighting what we have found to be the most important aspects for building a theoretical framework for eigentheory.

Thomas and Stewart (2011) found that students struggle to coordinate the two different mathematical processes (*matrix* multiplication versus *scalar* multiplication) captured in the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to make sense of equality as "yielding the same result" between mathematical entities (i.e., two equivalent vectors), an interpretation that is nontrivial or even novel to students (Henderson, Rasmussen, Sweeney, Wawro, & Zandieh, 2010). Furthermore, students have to keep track of multiple mathematical entities (matrices, vectors, and scalars) when working on eigentheory problems, all of which can be symbolized similarly. For instance, the zero in $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ refers to the zero vector, whereas the zero in $\det(A - \lambda I) = 0$ is the number zero. This complexity of coordinating mathematical entities and their symbolization is something students have to grapple with when studying eigentheory.

Thomas and Stewart (2011) also posit that this complexity may prevent students from making the symbolic progression from $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ through the introduction of the identity matrix, which is often an important step in solving for the eigenvalues and eigenvectors of a matrix A. In their genetic decomposition of eigentheory concepts, Salgado and Trigueros (2015) also point out the importance of understanding the equivalence of the two equations through a coordination of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and solutions to homogeneous systems of equations. Harel (2000) posits that the interpretation of "solution" in this setting, the set of all vectors $\boldsymbol{x}$ that make the equation true, entails a new level of complexity beyond solving equations such as $cx = d$, where $c, x,$ and d are real numbers. Our own work indicates student reasoning when solving eigentheory problems may be influenced by their reliance on or preference for one of the two eigen-equations (Watson, Wawro, Zandieh, & Kerrigan, 2017).

Hillel (2000) found that instructors often move between geometric, algebraic, and abstract modes of description without explicitly alerting students; although the various ways to think about and symbolize linear algebra ideas are second nature to experts, they often are not within the cognitive reach of students. In fact, Thomas and Stewart (2011) mentioned that students in their study primarily thought of eigenvectors and eigenvalues symbolically and were confident in matrix-oriented algebraic procedures, but "the vast majority had no geometric, embodied world view of eigenvectors or eigenvalues … losing out on the geometric notion of invariance of direction" (p. 294). In contrast, other researchers have shown how exploration of eigentheory through dynamic geometry software (Çağlayan, 2015; Gol Tabaghi & Sinclair, 2013; Nyman, Lapp, St John, & Berry, 2010), stretching geometric figures by a linear transformation (Zandieh, Wawro, & Rasmussen, 2017), gesture, time, and space (Sinclair & Gol Tabaghi, 2010), or real-world contexts (Beltrán-Meneu et al., 2016; Salgado & Trigueros, 2015) can be beneficial to developing conceptual understanding of eigentheory. We similarly agree on the importance of understanding eigentheory concepts in multiple ways and successfully navigating between these various modes of description.

Methods

The data for this study come from student written responses to the 6-question Eigentheory Multiple-Choice Extended (MCE) Assessment Instrument (Watson et al., 2017). This MCE aims to capture nuances of students' conceptual understanding of eigentheory and to inform our working framework of what it might mean to have a deep understanding of eigentheory. This work is part of a larger study of student understanding of eigentheory in mathematics and physics. However, this paper focuses on data from one sophomore-level introductory linear algebra class, at a university in the eastern United States. For this paper we focus on student responses to Questions 3 and 5 (Q3 and Q5), which are about linear combinations of eigenvectors (Figure 1). Of the 28 students in this class, 27 answered Q3 and 23 answered Q5. For each, students selected an answer to the multiple-choice stem and then were to respond to the open-ended prompt: "Because…(Please write a thorough justification for your choice)."

3. Suppose A is a $n \times n$ matrix, and $\boldsymbol{y}$ and $\boldsymbol{z}$ are linearly independent eigenvectors of A with corresponding eigenvalue 2. Let $\boldsymbol{v} = 5\boldsymbol{y} + 5\boldsymbol{z}$. Is $\boldsymbol{v}$ an eigenvector of A?
   (a) Yes, $\boldsymbol{v}$ is an eigenvector of A with eigenvalue 2.
   (b) Yes, $\boldsymbol{v}$ is an eigenvector of A with eigenvalue 5.
   (c) No, $\boldsymbol{v}$ is not an eigenvector of A.

5. Suppose a 3x3 matrix B has two real eigenvalues: for eigenvalue 2 its eigenspace E_2 is one-dimensional, and for eigenvalue 4 its eigenspace E_4 is two-dimensional. Also suppose that vector $\boldsymbol{x} \in \mathbb{R}^3$ lies on the plane created by the eigenspace E_4 and $\boldsymbol{y} \in \mathbb{R}^3$ lies on the line created by the eigenspace E_2, as illustrated in the graph below. If $\boldsymbol{z} = \boldsymbol{y} + 0.5\boldsymbol{x}$, which of the following is true?
   (a) The vector $\boldsymbol{z}$ is an eigenvector of B with an eigenvalue of _____ [fill in the blank]
   (b) The vector $\boldsymbol{z}$ is not an eigenvector of B.

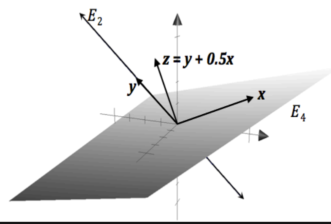

Figure 1. Questions 3 and 5 of the Eigentheory MCE Assessment Instrument.

Using Grounded Theory (Glaser & Strauss, 1967), each author of this paper open coded the students' open-ended responses to Q3 independently and discussed our results as a team to find interesting emerging themes. We repeated this process for Q5. In addition, we began comparing a student's open-ended response to Q3 with their response to Q5 to see if the pair of responses provided further insight into each student's understanding. Some of the themes that have emerged from our initial analysis are reported in the following section.

Results

We detail three preliminary themes from our analysis of justifications that students provided to support their conclusions on Q3 and Q5: (a) using the phrase "is a linear combination of" to support *both* correct and incorrect answers; (b) conflating scalars in a linear combination with eigenvalues, and (c) reasoning about *dimension* of eigenspaces versus *number* of eigenvectors.

Reasoning about linear combinations

In Q3, we noticed that 13 students wrote "$\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$" in their open-ended justification; however, 3 used it to support (a), 3 used it to support (b), and 7 used it to support (c). We note that the phrase "$\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$" was not written anywhere in the Q3 prompt; rather, the symbolic expression "$\boldsymbol{v} = 5\boldsymbol{y} + 5\boldsymbol{z}$" was given. We find it notable that so many students expressed this algebraic relationship in words and that this correct phrase was used to support all three solution options. Below we provide a few examples of responses supporting each solution option. Students are identified using labels of the form B#.

Examples of justifications given to support the correct solution (a), that $\boldsymbol{v}$ is an eigenvector with eigenvalue 2, are: "$\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$ which have the same eigenvalue"

[B72], and "$\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$. Since the value 2 already causes $\boldsymbol{y}$ and $\boldsymbol{z}$ to equal zero, adding a multiple to it will not change that" [B66]. We note that B72's response includes the critical information that $\boldsymbol{y}$ and $\boldsymbol{z}$ have the same eigenvalue – if this were not true, $\boldsymbol{v}$ would not be an eigenvector of A. It is not clear to us what B66 meant by his/her response, but we conjecture that it involved reasoning about solutions to the equations $(A-2I)\boldsymbol{y}=\boldsymbol{0}$ and $(A-2I)\boldsymbol{z}=\boldsymbol{0}$. In fact, in Watson et al. (2017) we highlighted B66, using data from work on other Eigentheory MCE questions, as an example of a student who showed some reliance on or preference for the homogeneous equation $(A-\lambda I)\boldsymbol{x}=\boldsymbol{0}$ rather than $A\boldsymbol{x}=\lambda\boldsymbol{x}$.

Examples of justifications given to support (b), that $\boldsymbol{v}$ is an eigenvector with eigenvalue 5, are: "$\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$. Both $5\boldsymbol{y}$ and $5\boldsymbol{z}$ are scalar multiples of their previous form so the resultant vector will be an eigenvector as well" [B71], and "Since it is a linear combination of the other eigenvectors, it would also be an eigenvector" [B69]. Note that 5 is the scalar associated with both $\boldsymbol{y}$ and $\boldsymbol{z}$ in the linear combination $\boldsymbol{v}=5\boldsymbol{y}+5\boldsymbol{z}$ given in the problem. However, 2 is the eigenvalue for both $\boldsymbol{y}$ and $\boldsymbol{z}$, and thus also for $5\boldsymbol{y}$, $5\boldsymbol{z}$ and $5\boldsymbol{y}+5\boldsymbol{z}$. The explanations given by each of these two students would be correct if they had circled the correct eigenvalue in the multiple-choice portion of the question. It may be that both B71 and B69 made a simple error in choosing 5 as the eigenvalue for v rather than the correct eigenvalue of 2; however, as we detail in the next subsection, it may be that these students conflated the scalar in the linear combination with the eigenvalue in a way more rooted in their thinking about what it means to be a linear combination of eigenvectors.

Examples of justifications given to support (c), that v is not an eigenvector, are: "Eigenvectors must be linearly independent from each other so if $\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$ then it cannot be an eigenvector" [B58], and "Because they all correspond to the same eigenvalue they all must have unique eigenvectors and $\boldsymbol{v}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{z}$ and therefore not unique and not an eigenvector of A." [B79]. One can understand how aspects of B58's reasoning were sensible to him/her, given that eigenvectors from *distinct* eigenvalues of a matrix are linearly independent. In addition it is common in textbooks to list a basis for the eigenspace as the solution to an eigenvector problem; this might lead students to believe that these linearly independent basis vectors are the only eigenvectors.

The phrase "is a linear combination of" was not as common in student responses to Q5. One notable exception is Student B69. This student answered Q3 correctly but gave the vague justification of "since it is a linear combination of the other eigenvectors, it would also be an eigenvector." On Q5, however, B69 explained that the vector would only be an eigenvector if the two vectors in the linear combination had the same eigenvalue (which is true). When considering B69's Q3 response in light of his/her Q5 response, we hypothesize that B69's vague response to Q3 was most likely based in a correct understanding of linear combinations of eigenvectors.

Conflating scalars in the linear combination with eigenvalues

We noticed that some student struggles with Q3 could possibly be explained by a conflation of the scalar 5 in the linear combination $\boldsymbol{v}=5\boldsymbol{y}+5\boldsymbol{z}$ with the scalar 2, which is stated as the eigenvalue for both $\boldsymbol{y}$ and $\boldsymbol{z}$. In addition to B71's justification that $\boldsymbol{v}$ is an eigenvector with eigenvalue 5 (seen in the previous subsection), consider B81's justification given to support (c):

> "No, because an eigenvector is defined as some linear combination defined by the eigenvalue so that $A\boldsymbol{x}=\lambda\boldsymbol{x}$, where $\boldsymbol{x}$ is the eigenvector and λ is the eigenvalue. The vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ are being scaled by a factor of 5 and $\lambda=2$ so they cannot be corresponding eigenvectors."

This student seems to conflate the scaling by 5 of the vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ in the linear combination with the scaling by 2 of the vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ when acted upon by the matrix A. In the former, $\boldsymbol{y}$ and $\boldsymbol{z}$ have not been acted upon by a transformation – the 5 is used to define the amount of each vector that is needed to create the vector $\boldsymbol{v}$. In the latter, the 2 is used to define that the result of multiplying each vector by A is twice the input vector. B81's reasoning seems to explain the role of the 5 in ways that would be more compatible with the role of the 2 and, because the scalars are different, concluded that "they" could not be eigenvectors. It is unclear what vectors are implied in the student's use of "they" – it could be some combination of $\boldsymbol{v}$, $5\boldsymbol{y}$ and/or $5\boldsymbol{z}$.

This preliminary result reminded us of another data set from our research group. In written data from final exams from an introductory linear algebra courses at a large public university in the southwestern United States, the instructor asked a question specifically targeting this potential conflation. The question first gave students a 3x3 matrix A and the eigenspace $k\begin{bmatrix}-1\\2\\2\end{bmatrix}$ and asked them to find the associated eigenvalue (the correct answer was -1). The question then asked students to complete the following and fill in the blank if appropriate: "The vector $\begin{bmatrix}-2\\4\\4\end{bmatrix}$ is … [an eigenvector of A with eigenvalue = ___] or [not an eigenvalue of A]." 13 of the 32 students correctly found that -1 was the eigenvalue for both vectors; however, 11 put that 2 was the eigenvalue for the second vector. Because it is two times the vector representative of the eigenspace, we hypothesize that these students conflated the scalar in the scalar multiple with the eigenvalue. These data need further investigation.

Reasoning about the dimension of eigenspaces versus the number of eigenvectors

Our third preliminary theme concerns students' reasoning about the possible number of eigenvectors in contrast to the dimension of the eigenspaces. On both Q3 and Q5, some student justifications referred to a finite number of eigenvectors; this is a potentially problematic view because each eigenspace has an infinite number of eigenvectors. For instance, on Q3, B78 reasoned that the linear combination of eigenvectors could not be another eigenvector because "Technically, you could multiply the eigenvectors by any number and if you did so and another eigenvector was achieved there would be a possibility for infinite eigenvectors which doesn't make sense." On Q5, reasons given by some students to support the correct choice (b) similarly focused on finite numbers of eigenvectors: "Matrix B already has 3 eigenvectors so there's no room for a 4$^{\text{th}}$" [B59], and "$\boldsymbol{z}$ is a linear combination of $\boldsymbol{y}$ and $\boldsymbol{x}$, and there are already 3 eigenvectors for 3 dimensions, so $\boldsymbol{z}$ cannot be an eigenvector of B" [B66]. We conjecture these students may have been conflating the total number of possible eigenvectors (infinite) for a 3x3 matrix with the number of linearly independent vectors (three) needed to create the bases for the one- and two-dimensional eigenspaces. Alternatively, B58's justification for Q5 focuses on dimension: "In a 3x3 matrix there can only be 3 dimensions to the eigenspace. E_2 and E_4 together span the entire space of $\mathbb{R}^3$ so there cannot be another eigenvector of B besides E_2 and E_4" [B58]. We conjecture grasping the difference between finiteness of dimensions and infiniteness of eigenvectors may be particularly important for understanding eigenspaces.

Discussion Questions

During our presentation we would like to discuss: how can we further investigate our three preliminary research themes, and what additional analyses might help us uncover students' creative and productive ways of reasoning about eigenspaces?

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