Developing Understanding of the Partial Derivative with a Physical Manipulative

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Multivariable calculus education is an area of growing investigation, and in this study we specifically target the topic of partial derivatives. Data was collected on students learning in an innovative curriculum using physical manipulatives. We trace the complex path as students developed both their mathematical knowledge and their use of the artifacts at their disposal, and analyze the interaction between them. Implications for the classroom and for research are noted.

Keywords: multivariable calculus, partial derivative, instrumental genesis, utilization scheme

Introduction & Literature Review

Multivariable calculus education has seen a surge of interest in recent years. A presidential panel emphasized it as a key course for the introduction of ideas that are complex and essential for STEM students (PCAST, 2012). Some research has been done on student conceptions in multivariable calculus of function (e.g. Martinez-Planell & Trigueros-Gaisman, 2012), derivative (Martinez-Planell, Trigueros-Gaisman & McGee, 2015), and integral (Jones & Dorko, 2015). Significant research has been done on student conceptions for single variable calculus, much of it in the last several decades, and some research has addressed how students construct multivariable calculus knowledge on those foundations (Dorko, 2016).

One central idea is that of rate, which underlies the derivative. During the typical curricular progression, students must develop an increasingly sophisticated conception of rate. They encounter a single constant rate with first linear single-variable functions, then a single nonconstant rate with nonlinear single-variable functions, then multiple nonconstant rates with multivariable functions. The first conception, including coordinating change in the input and output variables into a quotient, has been shown to be essential to developing the second conception (Pustejovsky, 1999; Zandieh, 1997).

The results reported here are from the implementation of an innovative curriculum designed to introduce important topics from multivariable calculus through student exploration with physical manipulatives. Students use, as representations of two-variable functions, surfaces which are molded from clear plastic and have a dry erase surface. Accompanying tools include an inclinometer (used to measure the slope at a point on the surface in a given direction), and domain mats (dry erase sheets with coordinate lines or contour lines). Students in small groups complete activity sheets in-class, which emphasize collaborative learning, student inquiry, and measurement with quantitative reasoning (for further details see Wangberg & Johnson, 2013). A previous investigation examined one aspect of how students learn about tangent plane and linear approximation using these tools (Fisher, Samuels & Wangberg, 2017).

Research Question

Even in light of the recent burst of research in multivariable calculus, little research has been done into how students develop conceptions of the partial derivative for functions of two variables. Further, for students who use physical tools to complete tasks and answer questions in their activities on multivariable calculus, little is understood about how they enhance their understanding of calculus, or what role the tools play in that process. At the nexus of these issues lies the following research question:

How do students develop conceptions of the partial derivative during exploration with a physical manipulative?

Theoretical framework

Verillon & Rabardel (1995) presented Rabardel's theory of instrumental genesis to explain the complex process by which a person engaged in achieving a goal adopts the use of some assisting object. The material object when first introduced is an artifact. For it to be a productive tool, the user must attach to the artifact a role in completing the present task. Actions and behaviors cognitively organized by the user for a class of situations comprise a utilization scheme. Schemes can be constructed personally by the user as derived schemes, or received in a social context as adopted schemes. The process of instrumental genesis produces an instrument, an artifact endowed with a set of utilization schemes for tasks, which is therefore a combination of material object and cognitive structures. During instrumental genesis, the artifact shapes the user through interactions which enhance the user's understanding of the subject matter, a process known as instrumentation. Additionally, the user shapes the artifact by developing utilization schemes for interacting with the artifact, a process known as instrumentalization. Thus, as user and instrument develop their partnership, each one causes a transformation in the other. Subsequent to the development of the theory, instrumental genesis was applied in mathematics education to understand student use of graphing calculators, computer algebra systems (Artigue, 2002), and dynamic geometry software (Leung & Chan & Lopez-Real, 2006).

Methodology

The data for this report were obtained from four students who worked as one group on an activity sheet designed to introduce the concept of the partial derivative. The first author, present as the instructor, asked questions to help make student thinking explicit and to encourage discussion and resolution of any disagreements within the group. The session was video recorded, the recording was transcribed, and the data were coded for instances of instrumental genesis by each author. Any differences of opinion were discussed until agreement was reached.

In the activity the students were tasked with measuring the partial derivative at a point on the surface using the inclinometer. The inclinometer used by the students had two rods, one round and one square with a bubble level attached, connected at the ends by a joint (see Figure 1).

Students could successfully complete the activity sheet with a utilization scheme for finding the partial derivative consisting of the following utilization schemes: one for the direction of the derivative, aligning the parts of the inclinometer in the proper vertical plane; one for the tangent line, positioning the round rod tangent to the surface at the selected point; one for representing the "run" (Δx or Δy in this context), positioning the square rod horizontal using the level; one for representing the "rise" (Δz), indicating a vertical displacement between the rods; and one for measuring change between two values for a variable, for which two possibilities are using a ruler or laying the inclinometer on the domain mat grid and counting boxes. As a result, they could calculate the quotient and find the partial derivative.

Figure 1. Student using the surface, inclinometer, and ruler over a domain mat.

The recorded session was split into episodes. Each episode consisted of discussion on approximately the same topic or in the same vein. Each episode was then coded with respect to utilization schemes. The students introduced two other utilization schemes in addition to the five schemes which comprised the partial derivative scheme described above: a scheme representing the normal line and a scheme measuring the interior angle of the inclinometer. Each scheme, when mentioned, was coded as attempted and completed (C), attempted with partial progress (P), or attempted with no progress (N), with an indication if it occurred specifically in the two-dimensional *y=f(x)* context (2).

Results

Here we present some results on the student work to find the partial derivative for a two-variable function at a given point, where the function is represented by a 3-dimensional plastic model. In the activity given to the students, the input variables *x* & *y* represented position, and the function value *T(x,y)* represented temperature. The following episode is presented with its coding and short description.

Interviewer: Okay, so, how would you use the same structure, the same orientation [as in the *y=f(x)* context], if you used it on the surface?
Student A: (put inclinometer on surface, round leg tangent, see Fig 2)
Student B: Like this?
Interviewer: Okay, so now describe to me what you are doing there.
Student A: Well I'm basically taking the point (lifting inclinometer and indicating the point), and I'm putting this on there, like, tangent.
Interviewer: Okay, so go ahead, do that.
Student A: (placing inclinometer tangent again) So this has to be...
Interviewer: Okay, so you got that one tangent.

Figure 2. Student A places the inclinometer tangent to the surface.

In the excerpt, the group had just transitioned from the two-dimensional, *y=f(x)*, context to the three dimensions, *z=f(x,y)*, context. The group, for the first time, demonstrated a complete utilization scheme for tangency, as well as directionality. However, with the square leg not horizontal, they did not implement the complete utilization scheme for Δy (which they had done before). They did not attempt to represent Δz, or to measure any displacements.

The coders identified 13 episodes in the recorded student activity, and the above description provides an example of how the coding was executed. The results are summarized in Table 1.

Discussion

Analyzing the students' actions through the lens of instrumental genesis gave valuable insight into describing both their struggles and achievements.

The Role of Instrumentation

In instrumentation, the artifact shapes the user through interactions which enhance the user's understanding of the subject matter. One example of successful instrumentation occurred in the episode below, in which students used the level to make the square rod parallel to the *xy*-plane to help represent displacements in the domain.

Student B: What is this for (indicating the level)?
Interviewer: Yeah, so what is that for?

Table 1. Utilization schemes and how they arose in each episode of the activity.

		EPISODE NUMBER												
	UTILIZATION SCHEME	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
Partial derivative schemes	**represent run**			C		C2		C2	N	C	N		C	C
	represent rise			C		C2							C	C
	measure change			C		C2	C						C	C
	directionality	C	C	C			C		C	N	N		C	C
	tangent line				P	C2		C2	C	C	C	C	C	C
Other schemes	**normal line**	C		C		C2	C							
	measure angle	P	P								P			

C: attempted and completed. P: attempted with partial progress. N: attempted with no progress. 2: in 2-dimensional *y=f(x)* context

Student C: To make it parallel.
Interviewer: That's to make sure that it's parallel. So why do you want it to be parallel?
Student B: *xy* parallel, right?

In this excerpt, the students observed a physical aspect of the artifact, the bubble level. Then, they proceeded to assign to it a functionality connected with mathematical content, being parallel to the *xy*-plane.

One early example of failed instrumentation occurred when the inclinometer shaped the thinking of one student in unproductive ways.

Interviewer: So tell me, tell me what you need to do, and then tell me how you're going to use the tool to do it.
Student B: First of all, the, perpendicular, the point, and we can find the right point, the bubble in the circle, and we can get the theta. So it means, this surface, this plane (indicating the *xy*-plane) and this plane (indicating the square leg) is parallel, so it means we can get this theta (indicating the angle, see Figure 3), and this theta is same. So-
Student C: -we can get-
Student B: -if we know theta...I don't know.

Figure 3. Student B indicates the angle on the inclinometer.

In this excerpt, the students focused on the angle presented by the tool. Using the angle was the very first idea suggested. One can hypothesize the reason, based on the appearance of the artifact. The primary physical characteristics of the artifact are two legs connected at their ends, and this mimics the standard instantiation of an angle. Further, the mobile joint allows for manipulation to any angle, which mimics the standard method to compare different angles. It is indeed possible to calculate the slope knowing the angle, however the students decided it was too complicated and did not pursue it. Thus, they did not devise a way to measure either the angle or the rate with this strategy, so this did not lead to new mathematical knowledge.

The Role of Instrumentalization

In instrumentalization, the user shapes the artifact by developing schemes for interacting with the artifact based in existing knowledge. One example of successful instrumentalization included using the inclinometer to represent the tangent line (as detailed in the first excerpt in the results section).

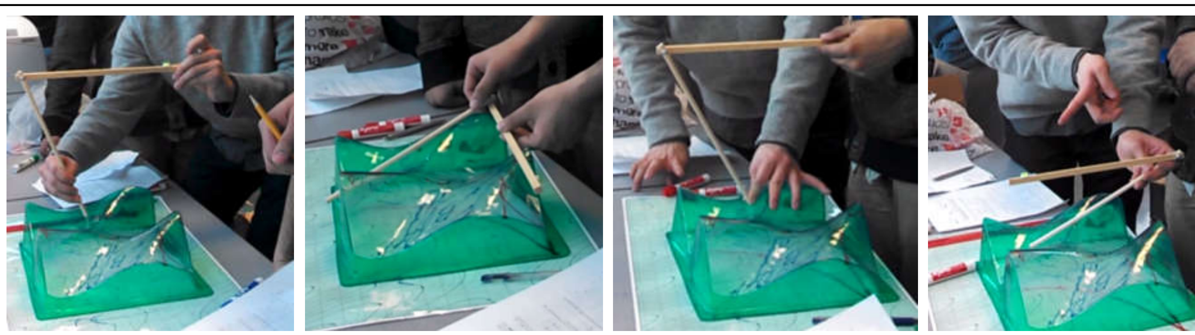
One example of failed instrumentalization included an attempt to measure Δz. The students previously demonstrated awareness of 3-dimensional rectangular coordinates, thus possessed the requisite mathematical knowledge, however the ruler was positioned between the ends of rods and not vertically (see Figure 4).

Figure 4. Student C measures the length between the rods' ends.

Extending Utilization Schemes from the 2-Dimensional *y=f(x)* to the 3-Dimensional *z=f(x,y)*

One recurrent unproductive move during attempts to find the slope was placing the inclinometer's round rod perpendicular to the surface for $z=f(x,y)$. One possible explanation is that the normal line is uniquely determined, while the tangent line is not. The students did not offer a justification for placing the inclinometer perpendicular to the surface, despite multiple prompts to do so. Yet when subsequently presented with the two-dimensional $y=f(x)$, they quickly placed the inclinometer tangent and found the derivative at a point.

Previous research has documented how the transition from single to multi-variable calculus presents significant challenges (Dorko, 2016; Jones & Dorko, 2015). The analysis with instrumental genesis revealed how students were grappling with an issue more pervasive for two variable functions, that of directionality. Early on in the activity, the student group lined up the entire inclinometer (both legs) in the appropriate direction (see Figure 5a). However, when forced to grapple with other considerations, particularly a utilization scheme for the tangent line by making the round leg tangent (while making the square leg parallel simultaneously), the complete directionality scheme was lost (see Figure 5b). Subsequently, the correct directionality scheme returned but the correct tangent line scheme again disappeared (see Figure 5c). Only then was the group able to merge all the correct schemes simultaneously to form the utilization scheme for finding the partial derivative (see Figure 5d).



(5a) (5b) (5c) (5d)

Fig 5. Students losing and regaining the utilization schemes for directionality and tangent line.

Equivalence between Utilization Schemes

One interesting deduction made by the students was the equivalence between certain utilization schemes. When calculating a derivative for $y=f(x)$, the students' utilization scheme to measure both the rise and the run consisted of laying the inclinometer flat on grid paper and counting boxes. When calculating a derivative for *z=f(x,y)*, their utilization scheme for measurement at one moment consisted of counting grid boxes. Later in the activity, it consisted of using a ruler. They used the results in equivalent fashion, referring to them in both cases as *dx* (or *dy, dz, dT*), and subsequently dividing the two numbers to calculate the rate. Members of the group were satisfied with both methods, and no one insisted on switching for either scenario.

The reasons for this difference in practice may come down to previous experience. For *y=f(x)*, by common instructional practice, students would have prior experience determining the slope of a straight line by counting grid boxes, and the inclinometer already lay on the grid paper when they reached this step. For *z=f(x,y)*, students in this class had prior experience determining the z-value (i.e. height) of a point using a ruler. Although the students were drawn to different utilization schemes in different contexts, their actions reflect that they used different schemes to get the same outcome.

Adaptation of a Previous Utilization Scheme

When students found the slope for *y=f(x)* using the inclinometer, they used what for them was a well-known process in a well-known scenario, but with an artifact they had only recently encountered. This was an example of adaptation of a previous utilization scheme. For the utilization scheme for tangent line on the surface, students initially used a pen as their tool to represent a tangent line on the surface. Subsequently, they used the (round leg of the) inclinometer to manifest the tangent line.

Another type of adaptation of a previous scheme occurred when the students moved from discussing derivative in the *y=f(x)* context to discussing it in the *z=f(x,y)* context. This progression actually occurred twice, first with the actual tools and second in discussion only. For *y=f(x)*, the students quickly and effectively created a utilization scheme to find the derivative. (As discussed previously, modifying the scheme for *z=f(x,y)* did not happen quickly or without struggle, either the first or second time.) In this case, the artifact was the same but the scenario had changed.

Utilization Schemes Disappear and Reappear

Students seemed to “forget” what they already knew, only to “remember” it subsequently. Students produced a utilization scheme for directionality almost immediately, and it stayed present in their manipulations and discussion for some time. However, in the process of grappling with certain obstacles that seemed to give them great difficulty, directional fidelity disappeared. The utilization scheme for tangent line took the longest to appear for the first time. Perhaps the group's greatest challenge was changing the relationship of the inclinometer round rod to the surface from normal to tangent. During the discussion before it happened, the group laid the inclinometer on the surface in new ways which ignored their previous directionality scheme.

One possible explanation is that the more challenging obstacles produced so much cognitive load (Sweller, 1988) that the students could not simultaneously consider or maintain directionality. Only after it was resolved, reducing cognitive load, could students return to considering their already-determined utilization scheme for directionality.

Instrumentalization during Development of a Utilization Scheme

Student mathematical knowledge and artifacts can interact in dynamic ways during instrumental genesis. At one point, Student A said “dee-T. Hold on, something over 12, I think. Dee-T should be 12. No, dee-T should be this one (measuring in the z-direction). Yeah, four.” He incorrectly coordinated the numerator and denominator in a slope calculation, before correcting himself. The apparent reason was that the triangle is upside down from typical usage, with the horizontal side higher in space than the vertical side (as in Figure 1). During the development of the scheme, the student confronted the signal from the inclinometer regarding the spatial relationship between ΔT and Δx (referred to by the student as dee-T and dee-x, respectively), initially accepting it before ultimately, and correctly, rejecting it. During this act of instrumentalization, it was necessary for the student to determine which information from the artifact to utilize (the lengths), and which to ignore (the relative positions of ΔT and Δx).

Linking Utilization Schemes

Students linked utilization schemes, to form what one might call a utilization super-scheme. They linked five schemes to form a scheme for finding the partial derivative. The formation of this linkage was clear when working on subsequent questions, and students found partial derivatives quickly, using the utilization scheme formed in the present activity.

Conclusion

Multivariable calculus is an essential course with numerous crucial ideas for students pursuing STEM. Innovation has been encouraged to improve learning and retention; concomitant with that is a need to analyze and understand student learning in these innovative contexts.

Implications for the Future

The students studied here had a tremendously difficult time generating a tangent line for a two-variable function. It was the last idea proposed and utilization scheme generated, and arose only after the interviewer introduced the *y=f(x)* context and scaffolded from there. Instructors might consider emphasizing tangent lines and planes early in multivariable derivative instruction to overcome this obstacle.

The transition from single variable calculus to multivariable calculus is one that students will continue to have to make, and one which presents considerable challenges. Previous studies considered the transition for the equation of a variable equal to a constant (Dorko, 2016) and integration (Jones & Dorko, 2015). In the current study, students needed to transition ideas such as tangent line and derivative. Further study of all aspects of this key transition is essential.

Summary

In this report we engaged in a study of four students in a group learning about the partial derivative through the use of a physical model while completing an activity sheet. We continued from previous work (Fisher, Samuels & Wangberg, 2017) our original approach to extend use of the instrumental genesis framework to contexts involving physical manipulatives. The use of the physical manipulatives illuminated the gaps in student knowledge, and also provided a path to fill them in. It is important to note that the data analyzed here covers the work of only four students, at a particular place and time, and we make no claims regarding generalizability to other students in other contexts. The students described here encountered numerous challenges as they extended their robust knowledge in the 2-dimensional *y=f(x)* context to the 3-dimensional *z=f(x,y)* context. They struggled to find a rate for a two-variable function, plumbing various parts of their mathematical knowledge while manipulating the artifacts at their disposal.

Reflected in their work were the role of instrumentation and instrumentalization as the students engaged in the mental constructions which turn the artifacts into tools. On this complex journey, the students devised utilization schemes, during which the student learning developed and manifested in noteworthy ways. This culminated in the development of the scheme for finding the partial derivative, which required coordinating the developed schemes. Thus, analyzing student actions through the lens of instrumental genesis proved effective and insightful to describe student learning activity in this context.

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