Exploring the secondary teaching of functions in relation to the learning of abstract algebra

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Secondary mathematics teachers regularly take advanced mathematics courses, but many regard them as unrelated to their work as teachers. In accord with a novel instructional approach (Wasserman et al., 2017), we designed materials for an abstract algebra course that connect to the teaching of functions in secondary schools. In this paper, we describe findings from a small-scale teaching experiment employing design research, which provides evidence that particular tasks were productive for accomplishing some of the mathematical and pedagogical aims.

Keywords: Functions, abstract algebra, secondary teacher education

Prospective secondary mathematics teachers (at least in the United States) are frequently required to take a large number of mathematics courses, including advanced courses such as abstract algebra and real analysis, to obtain certification to teach secondary mathematics. This is ostensibly with good reason – much of the content in secondary school is connected to and can be informed by ideas studied in these advanced mathematics courses (e.g., CBMS, 2012). Yet secondary teachers regularly report that completing such courses provides little professional value and does not influence their subsequent instruction (e.g., Zazkis & Leikin, 2010). This raises the challenging problem of designing tasks and modules in advanced mathematics courses that make meaningful connections to secondary mathematics teachers' future professional work.

In this paper, we explore the findings from a small-scale teaching experiment with two students in a secondary mathematics teacher education program. These students engaged with materials designed in accord with a novel instructional model for teaching advanced mathematics courses to secondary teachers (Wasserman et al., 2017). In particular, this paper looks specifically at their engagement with and reflections on abstract algebra content in relation to the teaching of functions in secondary school.

Literature and Theoretical Perspective

Advanced Mathematics Courses in relation to Secondary Teaching

Given the strong connection between ideas studied in advanced mathematics courses and the content of school mathematics, one would expect such courses to have an influence on secondary teaching. Yet findings from various studies appear to indicate the opposite. Monk (1994) examined the relationship between the number of university mathematics courses that a teacher completed and the learning outcomes of their students. The key finding was that courses beyond a fifth course – i.e., an advanced mathematics course – had little to no effect on the learning outcomes of that teacher's students. Zazkis and Leikin (2010) found that, according to practicing secondary teachers' self-reports, knowledge of advanced mathematics was rarely used and had little direct influence on their classroom practices. Other studies have reported similar results (e.g. Goulding et al., 2003; Rhoads, 2014; Wasserman et al., 2015).

This disconnect brings up the challenge of how to leverage the content of advanced mathematics in ways that are relevant to secondary teachers. Distinguishing between connections to the content of secondary mathematics and connections to the teaching of secondary mathematics, Wasserman (2016) analyzed school mathematics standards (CCSS-M, 2010) to identify four areas – arithmetic properties, inverses, structure of sets, and solving equations – where knowledge of abstract algebra might influence school mathematics' instruction. In general, the connections explored by Wasserman (2016) were specific to abstract algebra. That is, they were regarding *abstract-algebra-specific-content*, such as a group. We highlight this as a means to distinguish such content from other content that would also be related to the study of abstract algebra, but not necessarily unique to the study of abstract algebra, such as a function. We consider this to be *non-abstract-algebra-specific-content*. These sorts of connections have been given less attention in the literature.

A Novel Instructional Model

From Wasserman et al.'s (2017) point of view, the belief that completing a course in advanced mathematics will improve prospective or practicing teachers' (PPTs) ability to teach secondary mathematics has been based on a traditional view of transfer from the cognitive psychology literature (e.g., Perkins & Salomon, 2002). More specifically, there is an assumption that as a byproduct of learning advanced mathematical content, PPTs will better understand secondary mathematics content and will consequently respond differently to instructional situations in the future – a tenuously presumed "trickle down" effect (Figure 1a). Given the notorious difficulties in achieving this type of transfer, it is less surprising that PPTs' experiences in abstract algebra (or other advanced mathematics) often does not influence their teaching.

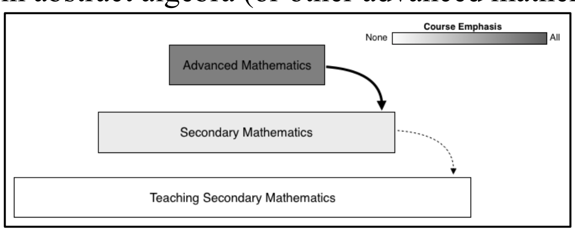

Figure 1a. Implicit instructional model for advanced mathematics courses designed for teachers

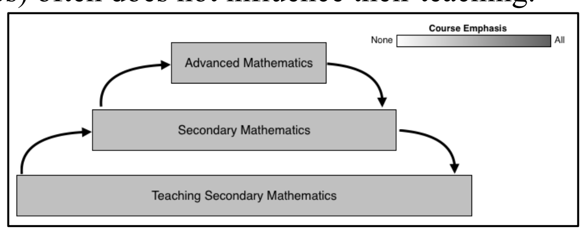

Figure 1b. Our instructional model for advanced mathematics courses designed for teachers

In Figure 1b, we present Wasserman et al.'s (2017) alternative instructional model for teaching advanced mathematics (including abstract algebra) in ways that can inform a PPT's pedagogical practice. This model is based on the premise that knowledge that PPTs learn should be inherently practice-based and applicable to the actual activity of teaching (e.g., Ball, Thames, & Phelps, 2008). Our model is composed of two parts: building up from and stepping down to practice. To *build up from (teaching) practice,* the abstract algebra content is preceded by a practical school-teaching situation. The building-up portion provides a context that sets the stage for the study of abstract algebra in ways that are both relevant to teachers' practices as well as well-suited to being learned in abstract algebra, which also aims to ease the challenges associated with transfer (e.g., Barnett & Ceci, 2002). The second part, *stepping down to (teaching) practice*, then uses the mathematical ideas from abstract algebra as a means to reconsider the secondary mathematics and relevant pedagogical situations. Stepping down to practice explicitly clarifies the intended mathematical and pedagogical aims of the abstract algebra content. In between building up from and stepping down to practice, the abstract algebra topics are covered by the instructor in ways true to its advanced nature with formal and rigorous treatment.

Methodology

In accord with Wasserman et al.'s (2017) instructional model, our research team designed five modules that were intended to connect content typically covered in an abstract algebra course – including binary operations, groups, isomorphisms, subgroups, and rings and fields – to various teaching situations. The modules included some of the *abstract-algebra-specific-content* connections identified by Wasserman (2016). For the purposes of this paper, however, we elaborate only on one module, the Functions Module, which leveraged the abstract algebra content of binary operations and isomorphisms as a means to converse, broadly, about functions and to reflect on the secondary teaching of functions. That is, the connection in this module was not about binary operations and isomorphisms per se, but instead used them as instantiations of and an opportunity to discuss functions – an example of a *non-abstract-algebra-specific-content* connection. Figure 2 gives an overview of this module.

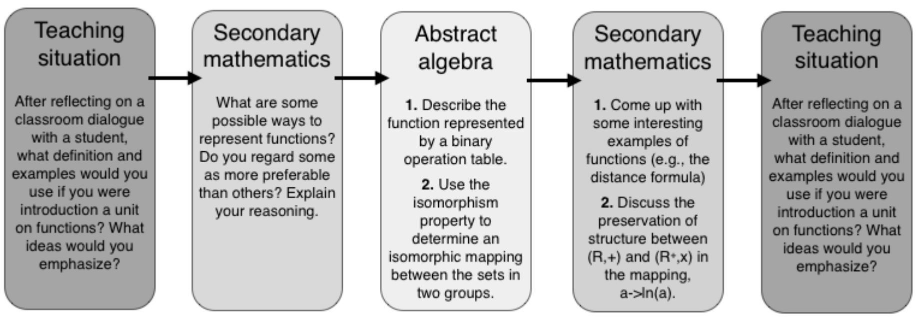

Figure 2. Overview of the Functions Module

Using design research (e.g., Cobb, et al., 2003) within a teaching experiment, the study engaged participants with some specific mathematical ideas and secondary mathematics teaching situations. Researcher-hypotheses were tested against participants' ways of thinking during the sessions. Two students (PPTs) enrolled in a program in secondary mathematics teacher education agreed to participate. One was a pre-service teacher, the other an in-service teacher with five years of experience (but not currently teaching). We collected and analyzed two sources of data: (i) a (transcribed) video-recording of PPTs engaging in the materials; and (ii) an (transcribed) audio-recording of a post-teaching-experiment semi-structured interview.

In our analysis, we compared what actually transpired during the teaching experiment to our hypothesized responses. First, we considered responses to the teaching situation. We characterized important aspects of PPTs' initial responses to the teaching situation, and their responses at the end of the module, and identified differences. Second, we considered whether these differences were in accord with our hypotheses, and, if not, whether they were meaningful instructional changes. Third, for each of these instructional differences, we then analyzed PPTs engagement with all facets of the module and their post-interviews to identify instances where their thinking appeared to shift in relation to the difference identified, and then to consider why this may have been the case. In this paper, we report on one instructional difference identified in the Functions Module that was in accord with our hypotheses, and discuss the aspects of the module that appeared to be most-closely associated with why PPTs responded differently.

Results

We organize the presentation of results from our analysis in terms of their support for two particular claims: 1) PPTs indicated their future teaching of functions would include novel mathematical examples and more non-mathematical examples, not just numerical ones; and 2) PPTs' struggle to view a binary operation table through a functional lens progressed through four stages and was productive for acquiring a deeper understanding of function, and was influential on their reported approaches to teaching the function concept to secondary students.

Claim 1

The first claim is that PPTs indicated their future secondary mathematics instruction would include novel mathematical examples and more non-mathematical examples of functions, not just numerical ones. As mentioned, this instructional change indicated by PPTs was essentially in accord with researcher hypotheses. We consider three sources of data in support of this claim: i) their initial reaction to the teaching situation; ii) their reflection back on the teaching situation at the end of the module; and iii) their interview responses after the module.

During PPTs initial discussions about a teaching situation (which is omitted here for the sake of space), they responded to the question: "If you were introducing a unit on functions, what definition and examples would you use? What ideas would you emphasize? Explain your reasoning." The definition they had mentioned already was that every input has a unique output. Their initial examples were pictorial mappings that demonstrated the idea of uniqueness with an example, $\{(1,1),(2,2),(3,3)\}$, and non-example, $\{(1,1),(2,2),(3,3),(3,2)\}$. Further examples included tables and graphs, and used a step function to reinforce uniqueness – that it was a function, but if you had two closed circles (at the same *x*-value) it would not be. They also included various other types of functions (linear, quadratic), using their equations to talk about inputs having unique outputs. The key point is that their initial examples of functions were numerical (i.e., in $R \times R$) – which was in accord with our hypotheses – and they used different representations of these kinds of functions to exemplify the issue of uniqueness.

After engaging with the material in the module, the PPTs reflected back on their responses. In contrast to functions with numerical inputs and outputs, their discussion focused almost exclusively on incorporating more abstract examples, especially real-world examples (e.g., people to birthdays, piano keys to notes). These examples emphasized the "mapping between two sets of objects" part of function in addition to the "uniqueness" part.

Interviewer: Uh, talk about, maybe some of the things you might do, uh, definitions and examples you might use or see with students.

A01: So, the birthday?

B03: Yeah, and I liked, I liked the piano, or anything that's, you know, not so mathy, I guess."

Interviewer: The birthday, piano, real world, so why that?

A01: I just think they help them connect, like, what the idea of a function is.

B03: Yeah, and sometimes I feel that in math, you have to do…

A01: Only numbers.

B03: Yeah, like add, subtract, multiply, and divide, yeah, numbers, and…where was I going with this? I don't know. There's this idea a function is outside of just add, subtract, multiply, and divide… it helps identify the idea that the function is just some mapping we describe by however we want.

During the interview after the module, we probed further into some of their thinking. Here, they mentioned part of the rationale for doing so was: "Just to give [students] other examples of things that are functions besides what we traditionally talk about in an algebra classroom." They also indicated, "I have other examples of things that are functions now that I didn't have before…And maybe some of these are too complicated to show them, but it would cause me to maybe stop and think about…maybe there's another mathematical thing that I could show them outside of the traditional $y = x + 3$… that is a function that's not normally something we would talk about as a function." One such example they considered including was the quadratic formula, i.e., the function, $(a,b,c) \rightarrow \left(\frac{-b+\sqrt{b^2-4ac}}{2a}, \frac{-b-\sqrt{b^2-4ac}}{2a}\right)$, which was one they had come up with previously during the module when asked to identify interesting examples of functions in secondary mathematics.

Claim 2

The Functions Module was designed to leverage two aspects of abstract algebra as a means the motivate discussions about function. The first was viewing a binary operation table through a functional lens; the second was leveraging isomorphisms to discuss an abstract example of a function mapping. As it turned out, the first was especially important for PPTs' reflections on secondary teaching – the second, less so.

During the post-interview, the PPTs singularly identified the binary operation table task – which is discussed in more detail below – as being particularly influential. Also, however, we briefly trace two other facets of the module that made their way into PPTs' responses: i) their mention of the "piano" example was connected to the function that was included in the module as a precursor to the isomorphism activity (but was not the isomorphic mapping itself); and ii) their "quadratic formula" example was one they identified during the module. Notably, for (ii), their initial reaction was that the quadratic function would *not* be a function because each (a, b, c) does not map to a unique output – there are two; later, they acknowledged it would be if the output set were pairs. This reiterates the idea that the objects being mapped to or from – and not just the idea of uniqueness – is important in determining whether something is a function, an aspect PPTs emphasized more readily in their teaching responses at the end of the module.

The primary activity in the module they identified as productive was the binary operation table task – where they were given the additive (mod 12) binary operation table and asked to "Describe the function (i.e., mapping) that this binary operation table represents." It was their (unexpected) struggle on this activity that appeared to have been especially productive for developing a deeper sense of function.

Interviewer: …So what were the main ideas that you got going through the abstract algebra content? …

B03: A deeper understanding of the function being something besides what I traditionally always thought about a mathematical function to be…

A01: I think that was the one that I had the hardest time—like the binary operation…

B03: And that one was really hard to think about cause it took us forever…it took us forever for us to figure out what the domain was.

A01: …it was a good place for us to get stuck.

B03: That's where I feel we, at least for me, I turned the corner about thinking about a function outside of just some linear situation… The fact that your domain can actually be an ordered pair…

We view their discussion here as indicative of a relationship between the binary operations task, which forced them to wrestle with and broaden their conception of function, and the real world and novel mathematical examples they mentioned including at the end of the module.

Since PPTs engagement in the binary operation task was profound, we looked further into reasons for why this may have been the case. In doing so, we identified four conceptual shifts that the PPTs went through as they came to view the additive (mod 12) binary operation as a function. During what we refer to as Stage 1, the PPTs had an *equation-view* of function. Their initial reactions to the task were:

A01: So, you're just, like, saying that $0 + 0 = 0$?

Teacher-researcher: Mmhm.

A01: Ok. …

B03: So, we just say, like, it's taking all the integers 0 to 11, and then…this is what we're mapping to?

A01: I don't know, I'm so confused… I'm so confused. I don't if this is the input, or…

B03: Are both of these inputs?

A01: … wouldn't these be outputs?... Unless it's like $x + y = z$. I don't know.

We point out that their initial, admittedly confused, attempts to view this as a function were by defining equations: $0 + 0 = 0$ and $x + y = z$. Now, these equations describe individual facts as well as more general truths about the binary operation table at hand. However, this equation-view was, ultimately, unproductive. For the next several minutes, the participants struggled to determine the domain – they cycled back and forth between thinking it was and was not "0 to 11." Their difficulties with the domain made their efforts to list actual elements in the mapping nearly impossible. The shift to Stage 2, a *mapping-view* of function, was facilitated by prompts to describe the mapping informally and to determine specific elements in the domain and range.

B03: It takes two of them… So it takes, it takes…if we do $A \times A$[1], we get all the ordered pairs, and then the added pairs get added together…like the two pieces of the pairs get added together to get that, but I don't know how we would write that.

A01: Ohhh.

Teacher-researcher: So… You don't have to be technical at this point. Just show me… Not just describe it, but show me things that map to things…

A01: $0 + 0$ maps to 0, $1 + 0$ maps to 1. It…

B03: Go all the way up to, like, $11 + 1$, and $11 + 1$ maps to 0.

Teacher-researcher: So what're you mapping? So what's the domain and what's the range?

A01: This [e.g., $0 + 0$] is our domain right? Cause this is being mapped to this [e.g., 0].

Although this may seem a trivial difference, we argue that viewing the binary operation table as $(0 + 0) \to 0$ and *not* $0 + 0 = 0$ was an important conceptual shift: it fostered their ability to identify, or at least get closer to identifying, elements in the domain and the range. The next shift, to Stage 3, a *multivariable-view* of function, was facilitated by the teacher stating that "the '+' is actually fairly irrelevant…" Their response was, "So we can just list the ordered pairs?... So now our domain is all these ordered pairs… And our range is over here." In other words, this shift allowed them to recognize the mapping as $(0, 0) \to 0$, which is more clearly indicative of the multivariable domain input and which removes the "+" from the domain. Last, the shift to Stage 4, a *dependent-view* of function, was facilitated by a student-teacher interaction.

Teacher-researcher: Ok. So the general set, the domain is what?...

B03: So $A \times A$ is the domain, and A is the range.

Teacher-researcher: … And so probably the easiest way to describe this is as a function is to say our function is taking things of the form here, it's taking two inputs, and it's mapping it to what? So if I have these inputs A and B, it's mapping it to…?

B03: $A + B$.

This last shift, guided by the instructor, was important, in that it allowed writing the mapping not as $(a, b) \to c$, but rather as $(a, b) \to (a + b)(mod\ 12)$. In other words, it established the element in the range set as being dependent on the input variables ("+" as part of the output, not input), which led easily to the participants recognizing the equation form of the function as: $f(a, b) = (a + b)(mod\ 12)$. These four stages appear to be conceptual shifts in PPTs' thinking on the binary operations task that facilitated their coming to a deeper understanding of function.

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[1] We note their use of the Cartesian product was likely prompted by the definition given, which was: A function ϕ mapping set A to set B is a relation between A and B (i.e., $\phi \subseteq A \times B$) such that each $x \in A$ is related to exactly one element in B.

Discussion and Conclusion

This study explored how PPTs engaged in and reflected on materials designed to connect the secondary teaching of functions to content in abstract algebra (binary operations and isomorphisms). The purpose was to explore materials with some non-abstract-algebra-specific-content connections that might be used in an abstract algebra course with (or for) secondary teachers. We discuss three points with regard to the primary claims from the findings.

First, the binary operation table task (much more so than the elaboration on isomorphisms) was productive for deepening PPTs' notions of function. It was during this activity that PPTs struggled, productively, to view a familiar operation through a functional lens. The four-stage process they went through lends some insights into the conceptual challenges they faced. Notably, these mirror, or perhaps elucidate, some shifts that secondary students also go through in their transition to understanding even more basic functions. Functional relationships are regularly introduced through an *equation-view* with two variables, e.g., $y = x + 3$. These then need to be understood as a *mapping*, $x \rightarrow y$, in particular between pairs of *numbers* (the multivariable stage was, essentially, about identifying objects in the domain and range), for which the *dependent* relationship between the two variables provides a more useful characterization of the mapping, $x \rightarrow x + 3$. This mapping, then, can be given the more formal and typical equation notation of a function: $f(x) = x + 3$. According to their own reports, engaging in this process with a more abstract example helped the PPTs recognize the broader ubiquity of functions, such as the quadratic formula mapping from 3-space to 2-space, the derivative relationship as a mapping between functions, etc.

Second, we present similarities and differences between the types of functions the PPTs indicated they might use as examples after the module – more abstract real-world examples (e.g., piano keys to notes) and novel secondary mathematics examples (e.g., quadratic formula). Both of these are more abstract, by which we mean that the sets being mapped to or from are typically not just a set of numbers, but rather a set of objects, letters, coordinate pairs, etc. However, there are some differences between real-world examples and novel secondary mathematics examples. First, real-world examples are already regularly introduced in secondary classrooms – but oftentimes only to be quickly discarded and forgotten. Given that PPTs' discussions valued looking at different representations of functions, including visual ones, this makes sense: real-world examples are harder to represent in multiple ways via tables, equations, graphs, etc. Now, "graphing" such real-world functions might in fact be an interesting exercise. However, in contrast, novel secondary mathematics examples such as the quadratic formula provide a similar sense of abstractness, but also may have the advantage of having other easily-identifiable representations to discuss (e.g., explicit formulas, graphs, etc.).

Third, we make a dual point about the PPTs' reflections on their teaching. On the one hand, the kinds of examples they ultimately discussed incorporating into their own teaching were in accord with the aims of the module. Indeed, one of the goals was that teachers should select examples that exemplify more nuances with the idea of function, which such abstract examples helped accomplish. On the other hand, many of the ideas came directly from the module materials or from their engagement with the module materials. Indeed, they even mentioned potentially having secondary students look at the (mod 12) binary operation task. We have seen this tendency before, of PPTs "transporting" materials from a teacher education setting, in the exact form they experienced them, to their teaching (Wasserman et al., under review). It exposes a tension in teacher education, and suggests that teacher educators may need to be more explicit about how general ideas (not just materials) might be adapted for and applied to teaching.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.

Barnett, S. M., & Ceci, S. J. (2002). When and where do we apply what we learn?: A taxonomy for far transfer. *Psychological Bulletin, 128*(4), 612.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9-13.

Common Core State Standards in Mathematics (CCSS-M). (2010). Retrieved from: http://www.corestandards.org/the-standards/mathematics

Conference Board of Mathematical Sciences (CBMS). (2012). *The mathematical education of teachers II.* Providence, RI and Washington, DC: American Mathematical Society and Mathematical Association of America.

Goulding, M., Hatch, G., & Rodd, M. (2003). Undergraduate mathematics experience: Its significance in secondary mathematics teacher preparation. *Journal of Mathematics Teacher Education, 6*(4), 361-393.

Monk, D. H. (1994). Subject area preparation of secondary mathematics and science teachers and student achievement. *Economics of Education Review, 13*(2), 125-145.

Perkins D., Salomon G., (2002). International Encyclopedia of Education, Transfer of Learning. 2nd ed. Oxford, UK: Pergamon Press.

Ticknor, C. S. (2012). Situated learning in an abstract algebra classroom. *Educational Studies in Mathematics, 81*(3), 307-323.

Wasserman, N. (2016). Abstract algebra for algebra teaching: Influencing school mathematics instruction. *Canadian Journal of Science Mathematics and Technology Education, 16*(1), 28-47.

Wasserman, N., Fukawa-Connelly, T., Villanueva, M., Mejia-Ramos, J. P., & Weber, K. (2017). Making real analysis relevant to secondary teachers: Building up from and stepping down to practice. *PRIMUS, 27*(6), 559-578.

Wasserman, N., Villanueva, M., Mejia-Ramos, J. P., & Weber, K. (2015). Secondary mathematics teachers' perceptions of real analysis in relation to their teaching practices. In T. Fukawa-Connelly, N. Infante, K. Keene, and M. Zandieh (Eds.), *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education (RUME)* (pp. 1037-1040). Pittsburgh, PA: RUME.

Wasserman, N., Weber, K., Villanueva, M., & Mejia-Ramos, J., P. (under review). Mathematics teachers' views about the limited utility of real analysis: A transport model hypothesis.

Zazkis, R., & Leikin, R. (2010). Advanced mathematical knowledge in teaching practice: Perceptions of secondary mathematics teachers. *Mathematical Thinking and Learning, 12*(4), 263-281.