Gauging College Mathematics Instructors' Knowledge of Student Thinking About Limits

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A theme in the literature surrounding instructional practices and knowledge for teaching is that knowledge of how students think about mathematical ideas plays important roles in supporting effective instruction. However, the undergraduate mathematics education community lacks tools for assessing this kind of knowledge. As an initial step toward the development of such assessments, we documented instructors as they examined students' work on calculus tasks during individual interviews. Transcripts were coded as exhibiting robust, limited, or no evidence of knowledge of student thinking using Jacobs, Lamb, and Philipp's (2010) framework. The coding process highlighted the varying depth and breadth of instructors' knowledge. Once refined, this coding process can be used to develop instruments for gauging knowledge of student thinking through means other than interviews. Such instruments will be of use to researchers, to those who design professional development for experienced and novice instructors, and for evaluation of professional development efforts.

Keywords: mathematical knowledge for teaching, limits, instructor professional development

Introduction

There have been many calls for increased attention to the teaching of undergraduate mathematics and professional development for those who do such teaching as part of efforts to improve enrollment and retention rates in STEM disciplines (Bok, 2013; Holdren & Lander, 2012). From extensive research at K-12 levels, we know multiple factors shape teachers' instructional practices (see, e.g., Borko & Putnam, 1996) and developing practices consistent with findings from research on teaching and learning can be challenging (see, e.g., Fennema & Scott Nelson, 1997). In this body of literature, a recurring theme is that knowledge of how students think about particular mathematical ideas plays important roles in supporting effective instruction (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Hill, Rowan, & Ball, 2005). Moreover, recent findings encourage increased active learning approaches in undergraduate instruction (Freeman, et al., 2014; Laursen, Hassi, Kogan, & Weston, 2014), and a necessary ingredient in enacting such instruction is a rich understanding of and ability to interpret student meanings. In particular, to respond effectively and support further student learning, instructors need to infer what thinking (correct or incorrect) might underlay what students write and say.

There are many ways in which teachers can and do develop such knowledge, including from experiences with their own students. Researchers of teacher professional development (PD) have touted the value of designing opportunities for teachers to examine and analyze students' written work (see, e.g., Little, Gearhart, Curry, & Kafka, 2003). College mathematics instructors frequently engage in this practice while grading homework or tests and/or while interacting with students who are working on problems as part of classroom activities. However, often the goal is to assess students' understanding rather than to unpack and make sense of that understanding.

Although knowledge of student thinking supports effective instruction and examining student work appears to be a context in which teachers learn how students think, the undergraduate mathematics education community lacks tools for assessing this kind of knowledge. Being able to gauge the depth and breadth of this knowledge for particular topics will aid both researchers (seeking to evaluate and/or study the development of such knowledge) and those who provide PD (who wish to determine the need for attention to particular topics). For example, such an assessment could inform PD activities so that they are accessible to the instructors based on their current knowledge of student thinking. This, in turn, could enhance the PD we provide to graduate students and other novice instructors of college mathematics.

The value of these kinds of assessment instruments is readily apparent from the body of work that was made possible at K-12 levels because of the existence of such tools (e.g., Hill et al., 2005). Such studies have been extremely powerful in establishing the importance of this form of knowledge in the K-12 mathematics community. Because of the content-specific nature of this knowledge, it is not possible to merely adopt existing instruments for use at the undergraduate level. As an initial step in this development process for topics in the undergraduate curriculum, we documented instructors as they examined students' written work on calculus tasks with the aim of developing a rubric for gauging the extent of their knowledge of student thinking. Our analysis was focused on answering two questions: (1) What meanings and/or ways of thinking do instructors attribute to the student work? (2) How extensive (or not) is each instructor's catalog of such meanings? Answers to these questions are needed to inform future development of short answer, multiple choice and/or case-based assessment items.

Research on Knowledge for Teaching

Findings from decades of research point to the important roles components of mathematical knowledge for teaching (MKT) play in teachers' practices and the learning opportunities they create for students. Of particular relevance to this study are the Pedagogical Content Knowledge (PCK) components of MKT that represent knowledge teachers use when hypothesizing what a student meant when they show their thinking when speaking, writing, and interacting with others. Studies have illuminated links between this kind of teaching-specific knowledge and both teachers' instructional practices and their students' learning (Carpenter et al., 1989; Hill et al., 2005). As part of these efforts, instruments have been developed to assess teachers' MKT. These instruments were based on findings from the substantial body of research on student thinking in the K-12 literature and on a robust set of classroom- and interview-based studies of teachers (Hill, Schilling, & Ball, 2004; Krauss, Neubrand, Blum, & Baumert, 2008).

As part of an effort to weave these findings from the K-12 MKT body of literature with findings from research on undergraduate student thinking, we are focused on categorizing college mathematics instructors' knowledge of student thinking. Our approach shares features with that of Jacobs, Lamb and Philipp (2010), who investigated the extent to which teachers paid "…attention to children's strategies but also interpretation of the mathematical understandings reflected in those strategies" (p. 184). By examining responses to assessments from teachers with varying levels of experience, these researchers were able to shed light on and characterize expertise in knowledge of student thinking, and document that this expertise can be developed.

Research Design and Methodology

We conducted our work from a cognitive theoretical perspective because of the prevalence of this perspective in the research on knowledge and knowledge development as well as the primarily individual nature of the out-of-classroom teaching work that is the focus of our investigation. This perspective, with the premise that human cognitive activity is accessible via written and spoken communication, has been used productively to examine teachers' knowledge and its roles in teaching practices (Borko & Putnam, 1996; Escudero & Sanchez, 2007;

Schoenfeld, 2007; Sherin, 2002). One specific way to access what teachers know about their students' thinking is by attending to what they notice when looking at student work (Jacobs, Lamb & Philipp, 2010). In this paper, we leverage Jacob, Lamb, and Phillip's noticing framework as a way to unpack what mathematicians know about their undergraduate students' mathematical thinking related to limit.

Interview data came from task-based individual interviews with seven research mathematicians at three institutions who had been recognized for their excellence in teaching, through being nominated for or winning a teaching award. The interviews were audio-recorded and transcribed to aid in the coding and data analysis. Tasks were taken from or modeled after tasks used in research on student thinking about limit, function (as it appears in calculus), and derivative. Interview design was adapted from one used previously to examine college instructor MKT (Speer & Frank, 2013). This consists of three parts per task for each interviewee: (1) Solve the task and describe the solution, (2) describe how students would solve the task, including difficulties they may encounter and/or mistakes they might make, and (3) examine and discuss student work, noting productive and unproductive ways of thinking demonstrated in each response. Here our focus is on the participants' insight and understanding of student thinking based on their examination of sample written work from the limit tasks.

Data analysis was guided by grounded theory (Corbin & Strauss, 2008) but also made use of findings from research on student thinking about limit, particularly those that provide insights into common productive and unproductive ways of thinking demonstrated by calculus students (e.g., Oehrtman, 2008, 2009). Following the approach of Jacobs, Lamb and Philipp (2010), chunks of interview data were labeled as demonstrating robust evidence, limited evidence, or absent of evidence of insight and understanding of student thinking. Descriptions of these levels were then developed considering similarities and differences among coded interview excerpts and among various levels of evidence. Our unit of analysis was each participant's discussion of the set of student responses to an individual task.

Findings

We found that Jacobs, Lamb, and Philipp's (2010) framework for classification of knowledge of student thinking was easily adapted to our data set. We demonstrate the applicability of this approach for use with instructors at the post-secondary level with interview excerpts and descriptions of how we operationalized the robust-limited-absent evidence rubric for use with our data. Although mathematicians examined several samples of student work, we illustrate our findings with data from their discussions of one student response to the prompt: "Describe what it means when we say 'the limit of $f(x)$ as x approaches 3 is 12' ($\lim_{x \to 3} f(x) = 12$)." The sample written student response said, "It looks like *f(x)* is 12 although *x* never actually reaches 3." Descriptions and transcript excerpts are shown in Table 1.

In interview excerpts coded as being *absent* in demonstrating insight into student thinking, the instructor does not explain what the student might be thinking and only notes that the response is not correct, remarking that she would assign it partial credit on a test. In the interview excerpt coded as *limited*, the instructor recognizes that this is a typical student response, suggesting one possible reason why a student might come up with this answer. However, this response is somewhat vague and does not demonstrate extensive or rich knowledge of student thinking. Responses coded as *robust* demonstrate an understanding of and validity in the student work, despite the not completely correct response. To demonstrate robust interpretation of student work and knowledge on this scale, one must provide examples of why students might answer in such a way, recognizing common productive/unproductive ways of thinking and describe possible origins for such ways of thinking.

Table 1. Examples demonstrating three levels of knowledge of student thinking

Level	Description	Example from Interview
Absent	Does not articulate understanding about student thinking when interpreting the student work. Further, does not consider the student perspective and see validity in the portions of students' incorrect responses.	**Interviewer:** If that was an answer you saw on a test, or someone said to you in office hours, how do you feel about that one? **Instructor 1:** (pause) I don't know. **Interviewer:** It has that approaches 3 thing that you mentioned before. **Instructor 1:** Yeah, there are some parts that I would give some partial credit, but I certainly would not get full credit because you know it is like "it *looks* like f(x) is 12", that is not quite correct, and "although *x* never reaches 3", so I would probably give something like partial credit, but no full credit.
Limited	Some articulation of student thinking; able to explain thinking demonstrated in common student responses, but either some pieces are left unexplained, or the interpretation of student work is vague.	**Instructor 2:** I mean this is – the very apt response, arcitypical [archetypal]. **Interviewer:** arcitypical? **Instructor 2:** No I think it is fine; … I think this is a reasonably good approach. A good answer, if imprecise. It looks like f(x) is 12, so that's – I mean, it is just lacking the language to say that f(x) is within epsilon-delta of 12, right? **Interviewer:** There is a tolerance – **Instructor 2:** One way to imagine is that this is accompanied by a graph. And maybe they are thinking of the graph picture.
Robust	Able to take on the student's perspective in all responses, including less common errors, finding validity in aspects and recognizing where the student is still lacking in complete understanding.	**Instructor 3:** That's fine, this is exactly Newton's way of approaching limits. **Interviewer:** What is the student thinking? **Instructor 3:** So that's exactly this picture of as x *approaches* 3, but never touches…the limit they learned that you don't necessarily hit, and Newton had exactly this issue of what happened when you were trying to calculate the derivative… **Interviewer:** Alright, and what do you make of the "it looks like f(x) is 12"? **Instructor 3:** That's a response to this stuff, limit of f(x)=12, that's a reinterpretation of what they would think there.

Discussion and Implications

The goal of this work was to categorize what instructors do when interpreting student meanings/thinking in order to gain a better understanding of what it means to have knowledge of student thinking of undergraduate mathematics topics. This is the first step to developing an assessment grounded in practice adapting Jacobs, Lamb, and Philipp's (2010) framework of classifying the professional noticing of teachers.

The coding process revealed that mathematicians demonstrated multiple levels in their interpretation of student work across various tasks. This suggests that even experienced instructors may have had different opportunities to hear students' reasons for answers or may engage differently with student work on different tasks or topics. Although the central goal of the analysis was to inform assessment development, we offer preliminary thoughts on this apparent variation. Differences could be related to their dispositions towards student thinking and willingness to engage with the student work (although we note that all participants appeared to engage with the tasks). Alternatively, they may have had different opportunities to engage with student thinking due to differences in their instructional approach (i.e., there may be fewer opportunities to hear student thinking in a lecture-based class than one utilizing collaborative groupwork). Further analysis is needed to test and refine this tool to gauge instructor knowledge. This will entail examining data from additional instructors (including more novice instructors) as they examined written student work on limit tasks and also an expansion to data we have from the same instructors as they examined student work from function, derivative and integral tasks.

By operationalizing the Jacobs, Lamb, and Philipp's (2010) framework for our limit tasks we begin the work of identifying characteristics of varying depths and breadths of knowledge of student thinking. We see two features as varying across the levels. One relates to the richness of the participants' descriptions of the student thinking. These range from non-existent to including multiple diagnoses for what a student might have been thinking. Participants also varied in the extent to which they articulate hypotheses for *why* responses might have seemed reasonable to students. We view this empathetic disposition as aligned with Smith III, diSessa, & Roschelle's (1994) perspective on student misconceptions and as an important avenue for further study.

Once refined, these characteristics can then be used to develop instruments for gauging this knowledge through means other than interviews. Such instruments with items in open response, multiple-choice or case-based formats would make other approaches to this work feasible, including assessment in larger instructor populations and/or coupled with observations of teaching. Such a tool could also be used to track growth in instructors' abilities to interpret student thinking and depth of knowledge over time and/or after participating in PD.

Instances of robust understanding provide us with evidence that mathematicians can develop such knowledge. We note that development of this knowledge most likely occurred from their on-the-job experiences of examining student work and interacting with students given the scarcity of teaching-specific professional development opportunities typically available for college instructors (Holdren & Lander, 2012). The varied levels demonstrated by participants suggest that examining student work could be productively used in college instructor PD to further enhance their practice-based learning opportunities. In addition, such an approach could provide an opportunity for exposure to not only student work/student thinking, but also help develop instructors' understanding that knowledge of student thinking is indeed a set of knowledge that is (1) desirable for instructors to possess to support effective instruction, and (2) something that can be attended to and enriched over time. Moreover, this leads to the possibility for more targeted PD when armed with insight into types of responses one might expect from experienced and novice instructors. It also has the potential to illuminate and document growth in knowledge of student thinking over time, pointing to the influence of PD or experience in this development. Utilizing such approaches in the professional development of undergraduate instructors can help our community improve the learning opportunities we create for students.

References

Bok, D. (2013). We Must Prepare Ph.D. Students for the Complicated Art of Teaching. *The Chronicle of Higher Education*, (November 11, 2013). Retrieved from http://chronicle.com/article/We-Must-Prepare-PhD-Students/142893/

Borko, H., & Putnam, R. T. (1996). Learning to Teach. In D. C. Berliner & R. C. Calfee (Eds.), *Handbook of Educational Psychology* (pp. 673–708). New York: Macmillan Library Reference USA: Simon & Schuster Macmillan.

Carpenter, T., Fennema, E., Peterson, P., Chiang, C., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal*, *26*(4), 499–531.

Corbin, J., & Strauss, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.). Thousand Oaks, CA: Sage Publications. http://doi.org/10.4135/9781452230153

Escudero, I., & Sanchez, V. (2007). How do domains of knowledge integrate into mathematics teachers' practice? *The Journal of Mathematical Behavior*, *26*(4), 312–327. http://doi.org/10.1016/j.jmathb.2007.11.002

Fennema, E., & Scott Nelson, B. (1997). Mathematics Teachers in Transition. *Studies in Mathematical Thinking and Learning*. Mahwah, New Jersey: Lawrence Erlbaum Associates.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences*, *111*(23), 8410–8415.

Hill, H., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, *42*(2), 371–406.

Hill, H., Schilling, S., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal*, *105*(1), 11–30.

Holdren, J. P., & Lander, E. S. (2012). *Engage to excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics*. Washington, D.C.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional Noticing of Children ' s Mathematical Thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202.

Krauss, S., Neubrand, M., Blum, W., & Baumert, J. (2008). The professional knowledge of German secondary mathematics teachers: Investigations in the context of the COACTIV project. In *11th International Congress on Mathematical Education*. Monterrey, Mexico.

Laursen, S. L., Hassi, M.-L., Kogan, M., & Weston, T. J. (2014). Benefits for Women and Men of Inquiry-Based Learning in College Mathematics: A Multi-Institution Study. *Journal for Research in Mathematics Education*, *45*(4), 406–418. http://doi.org/10.5951/jresematheduc.45.4.0406

Little, J. W., Gearhart, M., Curry, M., & Kafka, J. (2003). Looking at student work for teacher learning, teacher community, and school reform. *Phi Delta Kappan*, *85*(3), 184–192.

Oehrtman, M. (2008). Layers of abstraction: Theory and design for the instruction of limit concepts. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics education, MAA Notes Volume 73* (pp. 65–80). Washington D.C.: Mathematical Association of America.

Oehrtman, M. (2009). Collapsing Dimensions, Physical Limitation, and Other Student

Metaphors for Limit Concepts. *Journal for Research in Mathematics Education*, *40*(4), 396–426.
Schoenfeld, A. H. (2007). Method. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 69–110). New York: MacMillan.
Sherin, M. (2002). When teaching becomes learning. *Cognition and Instruction*, *20*(2), 119–150.
Smith III, J. P., diSessa, A., & Roschelle, J. (1994). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *The Journal of the Learning Sciencess*, *3*(2), 115–163.
Speer, N., & Frank, B. (2013). Building knowledge for teaching rates of change: Three cases of physics graduate students. In S. Brown, G. Karakok, K. H. Roh, & M. Oehrtman (Eds.), *Proceedings of the 16th Annual Conference on Research in Undergraduate Mathematics Education, Vol. 2* (pp. 247–253). Denver, CO.