Empowering Students in Learning Proof: Leveraging the Instructor's Authority

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When students are coming to understand how to construct proofs, as well as how mathematicians use proofs in their work, the role of the instructor cannot be overstated. In this paper, we present an investigation into how an instructor uses her authority to empower students as legitimate proof producers and learners of mathematics. We view this empowerment and student learning through a situated lens, accounting for relationships of disciplinary authority and student agency. In our investigation, we analyzed the transcripts from three classroom episodes in an inquiry-based transition-to-proofs course. We identified instances when the instructor leveraged her institutional authority as well as her mathematics expertise authority to support students' engagement in the dance of agency, asserting their own creative ideas as learners of mathematics while still adhering to the norms and standards of the discipline.

Keywords: Authority, Agency, Proving, Proof, Empowerment

Mathematical proof has a special place in the discipline of mathematics, and it plays an important role in undergraduate mathematics education (Stylianides, Stylianides, and Weber, 2017). When students learn to write proofs, they must find a balance between the norms of proof writing established by the discipline of mathematics and their own creativity. Such a balance may be considered a dance of agency between the authority of the discipline and the agency of the student (Boaler, 2002; Pickering, 1995). The instructor plays an important role in setting the stage for this dance, with his or her position to both represent the discipline and support students' agency. In mathematics education literature, the problem of striking a balance between the norms of the discipline and the ideas of the students is often framed as a problem of establishing shared authority between student and teacher in the classroom (e.g. Webel, 2010). So how can shared authority be achieved? Gerson and Bateman (2010) suggested one potential answer--that undergraduate instructors should limit both their *institutional authority* (based on their position as instructor of the course) and their *mathematical expertise authority* (based on proven mathematical expertise) in order to promote shared authority within a classroom community. But, just as students must find a balance between the authority of the discipline and their own agency, we contend that instructors must find a balance between exerting and limiting their own authority in order to facilitate productive learning. In fact, we believe the field could benefit from questioning the often-cited claim that teachers should *limit* their authority, and instead explore how instructor authority can usefully be employed to support students as mathematics learners. In this study, we use data from an inquiry-based undergraduate transition-to-proof course in order to understand how an instructor may use her institutional and mathematical expertise authority to empower students as legitimate practitioners of mathematics.

Theoretical Framework

Situated Learning

We adopt a situated view of learning and consider learning to occur when students engage as legitimate peripheral participants in the mathematics community of practice (Lave & Wenger, 1991; Wenger, 1998). "Viewing learning as legitimate peripheral participation means that learning is not merely a condition for membership, but is itself an evolving form of membership" (Lave & Wenger, 1991, p. 53). So in our case, to learn mathematics is to be in an evolving state of membership within the mathematics community. Students, as "newcomers" to the mathematics community, gradually develop their agency and shape the discipline itself as they become more accustomed to the norms practiced by the "oldtimers" of the discipline (Lave & Wenger, 1991). Through this process, a student gains fuller membership into the community of mathematicians. Thus, understanding how to facilitate the balance between the authority of the discipline and the agency of the learner is critical to supporting students as learners of proof, and hence legitimate practitioners of mathematics.

Agency and Authority

Recall that in this study, we seek to understand how instructors may use institutional and mathematical expertise authority to empower students as legitimate practitioners of mathematics. Empowerment, as we conceive it, is linked to a student's ability to engage in the *dance of agency* (Boaler, 2002; Pickering, 1995), balancing his or her own creative ideas with the ideas and norms of the discipline. Although Boaler (2002) and Pickering (1995) use the term *agency* to describe both the individual and the discipline, we find it more useful to follow Amit and Fried (2005) and speak of the *agency* of the student and the *authority* of the discipline. A few definitions will be helpful in this regard. By *agency*, we mean "a dynamic competence of human beings to act independently and to make choices" (Andersson and Norén, 2011, p. 1). Given this definition, agency is most appropriate when describing a human being in action. We identify instances of student agency when students debate, ask questions, and propose their own ideas or proving strategies.

Alternatively, by *authority*, we refer to "situations in which a person or group, fulfilling some purpose, project or need, requires guidance or direction from a source outside himself or itself…. The individual or group grants obedience to another person or group (or to a rule, a set of rules, a way of coping, or a method) which claims effectiveness in mediating the field of conduct or belief as a condition of receiving assistance" (Benne, 1970, pp. 392-393). In the undergraduate setting, students experience the authority of the discipline in textbooks and communications from their instructor. Established theorems, standards for rigor, symbolic conventions, and accepted proof formats are instantiations of the authority of the discipline of mathematics. We contend that the discipline possesses authority in terms of its norms and standards for rigor and proving.

The instructor plays an important role in not only promoting student agency (e.g., by asking probing questions, providing opportunities for discussion) but also in helping students learn about what is established in the discipline of mathematics. In this sense, the instructor serves as a broker between the community of students in the classroom and the community of mathematicians in the discipline (Wenger, 1998). If we consider a continuum to represent students' learning of proof, as in Figure 1, in which the right side of the continuum represents students exerting their agency in deciding what should count as mathematical proof, and the left side represents obedience to disciplinary authority, we recognize the limitations of each extreme. If students are strictly taught to obey the authority of the discipline in regards to learning mathematical proof, then they may focus on form rather than reasoning. They may fail to see proof as a sense-making endeavor and feel isolated from the process. On the other hand, if students are given complete agency to decide what counts as proof, their proofs may not meet the standards of the discipline. This makes the instructor's role as broker between the classroom community and mathematics community central to empowering students in the dance of agency.

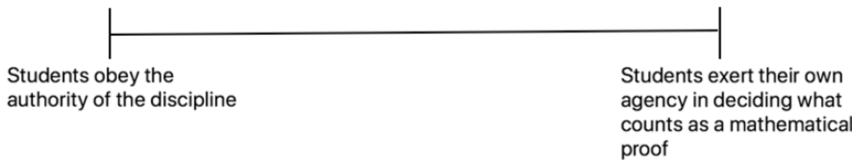

Figure 1. Continuum of Agency and Authority when Learning Proof

Methodology

We adopted Gerson and Batemen's (2010) classification of authority, because to date they have provided the most extensive framework for identifying authority roles within mathematics classrooms that "give way to the realization of autonomy with interdependence" (p. 195), a goal we see as aligned to our theoretical perspective, in which we view student empowerment as a successful balance of student agency and the authority of the discipline of mathematics. As a way to describe the interactions amongst students and the instructor in an inquiry-based calculus class, Gerson and Bateman distinguished between four authority types (hierarchal, mathematical, expertise, and performative) encompassing seven sub-types. Two of these subtypes are particularly important to our study: *institutional authority* which is "held by instructors based on the position as instructor of the course" (p. 201), and mathematics expertise authority, "based on the proven mathematical expertise of the bearer" (p. 201). Recall, Gerson and Bateman suggested that instructors should limit these two sub-types of authority in order to promote shared authority in the classroom. We, however, sought to explore if and how an instructor's use of these two authority sub-types may support student learning, highlighting the teacher's role as a more nuanced set of actions and decisions. Hence, we pose the following research questions: Can an instructor's *institutional authority* be used to empower students as legitimate practitioners of mathematics? If so, how? Can an instructor's *mathematics expertise authority* be used to empower students as legitimate practitioners of mathematics? If so, how?

The data for this study were collected as part of a larger study on the nature of mathematics (Pair, 2017). For the purpose of this study, we transcribed audio recordings of three classroom episodes depicting whole-class discussions that took place within an undergraduate transition-to-proof course at a large Southeastern University. Two mathematics education scholars, also authors of this manuscript, co-taught the course utilizing collaborative, inquiry-based pedagogy. Twenty-three students, including mathematics majors and minors, agreed to participate.

Three researchers independently read and coded the selected transcripts, taking one speaker's (student or instructor) turn as a unit of analysis. Each time a new speaker contributed to the whole-class discussion, the researchers assigned as a code any of the seven authority sub-types from Gerson and Bateman's framework that seemed to represent the authority relationship demonstrated within that speaker's contribution. Sometimes a researcher assigned more than one authority code to a turn, and sometimes a researcher assigned no codes to a turn. Also, researchers wrote memos and questions related to their coding in the margins of the transcript. After independent coding of each transcript, the three researchers came together in group meetings to (a) discuss and negotiate the types of authority relationships evident in the transcript, and (b) identify key instances within the transcript where *institutional authority* and/or *mathematics expertise authority* were used by the instructor in a way that empowered students as legitimate practitioners of mathematics.

Results

To begin our presentation of results, we first reiterate what we mean by *empowered learning* with respect to students as learners of proof. Recall, we view empowerment as related to a student's ability to exert his or her own agency/ideas while also considering and respecting the norms and understandings of the discipline of mathematics. To clarify this notion, we present two student quotations from an early class session in the transition-to-proofs course. Students were asked to respond to the following question, "Based on your past learning experience with mathematical proof (either high school or college), *how did you learn* about what makes a good mathematical proof?" We highlight two responses below, one from Eddie and one from Josiah.

> And as far as my personal struggles with them [proofs]... what helped me the most honestly was continually being wrong. I would be wrong and the teacher would be like this doesn't work. Logically this doesn't work, that math is wrong there. And so the more I was wrong and the more I thought about why I was wrong and how to fix it, I got better at it. (Eddie)

> I feel like I haven't really formally learned about how to do proofs. In the lower classes it was something that was just sort of tacked in like in the textbook, as like oh here is a thing, you understand this right? Or let's look at how to do integration with this infinite series, and everyone is like WHAOOO??? And by the time you get to the higher classes they just assume you know how do to it. (Josiah)

We argue that the first student, Eddie, describes an empowering learning experience because he is actively engaged in the dance of agency. Eddie exerted his ideas (agency), and was often incorrect. He would receive feedback from his teacher (representing disciplinary authority) and then take the feedback to revise and reflect on how to modify his proofs, in an empowering cycle. Alternatively, Josiah describes a disempowering learning experience as he did not have the opportunity to engage in the dance of agency. He was instead simply given the instantiation of authority from the discipline (in the textbook), or assumed to understand the knowledge and norms of the discipline, with little opportunity for inserting his own agency.

With this conceptualization of empowerment in mind, we turn to our research questions: (1) Can an instructor's *institutional authority* be used to empower students as legitimate practitioners of mathematics? If so, how?, and (2) Can an instructor's *mathematics expertise authority* be used to empower students as legitimate practitioners of mathematics? If so, how?

Leveraging Institutional Authority

Early on in the semester, students sought direct guidance from the instructors regarding what constitutes mathematical proof, asking questions including, "Is there a best way of doing proofs? Something that works stronger than all other ways?", "What is official/professional proof supposed to look like? What are the requirements?," and "When will we know for sure we are writing proofs correctly?" Instead of providing a direct answer about what makes a proof, the instructors engaged the students in a two-day activity in which students debated the criteria for a valid argument by analyzing and critiquing one another's written arguments for a given problem.

The quotation below comes from the lead instructor, Dr. BB, as she responded to students' questions regarding the best way of writing proofs:

> We are looking for some more guidance right? About what is proof. You all were working last week, working on a few problems and thinking about how to prove things. Today what we are going to do, is try to as a class, as a community, come up with some criteria that would help us describe what a proof should be. Okay? And we [the instructors] think that can come from you all, that it doesn't necessarily need to come from us. That based on logic and based on the understanding of mathematics that you have so far, that you all are very capable of creating some criteria that would help you think about what should count as proof and what maybe shouldn't count as proof. Okay? So that is what we are going to do today.

Creating a class criterion for proof writing represents students interacting at the rightmost end of our continuum (Figure 1); students exerted their own agency on what counts as correctness in proof. This exercise was a novel experience for most students, as the majority of students described learning mathematical proof in ways similar to Josiah, where proofs were given as examples to be mimicked or memorized. We noticed that in these early-semester encounters, the instructors often leveraged their *institutional authority* in order to drive the norms of the classroom, asserting that students could and should insert their voice into the classroom conversation, and implementing a classroom activity that allowed for active student contributions. Given students' initial inclination to live on the left side of the continuum (obeying disciplinary authority), we argue that the instructors' use of institutional authority here was empowering as it allowed students the opportunity to offer their own thoughts and ideas to the classroom discourse. This is highlighted again in the following exchange, as Jackson contrasted his classmates' ideas and his own thoughts on proof:

> *Jackson:* I'm having a debate about ... examples about where they belong in proofs. I guess I may not agree with the [class] consensus that examples belong in proofs. I certainly do examples for myself to support the veracity of what I am working on. But I don't think a thousand examples prove anything. So I don't know if they belong in there or not. I think one example that disproves has a lot of value. But I don't know if putting half a dozen examples in a proof really supports the proof or not.
>
> *Dr. BB:* Mhhm. Great. And [that's] another thing that I would like all of us to be thinking about today, okay? So keep raising that question Jackson, when we get back to revising our criteria later maybe you could bring it up again based on what you discuss in your group today.

This instance can be interpreted both as a leveraging of institutional authority and as a limiting of mathematics expertise authority. Dr. BB leveraged her institutional authority (as the instructor of the course) by encouraging Jackson and the other students in the class to continue to think about the role of examples in proofs. She simultaneously limited her mathematics expertise authority, as Gerson and Bateman (2010) suggest, by not offering her own thoughts (as a disciplinary expert) on the role of examples in proofs. Dr. BB's actions continued to emphasize the norm that student contributions were valuable to this classroom community, and these actions provided opportunities for empowered learning by moving students toward the right side of the continuum. Note that even though students were exerting their agency, they were coming to conclusions that were aligned to the disciplinary norms of the mathematics community. While a mathematician may use examples to generate ideas for a proof (de Villiers, 2004) or check the claims of a proof (Weber, Inglis, & Mejia-Ramos, 2014), examples are not part of formal deductive argumentation. Eventually, the class was able to come to this conclusion on their own through whole-class negotiation. We observed several similar instances early in the semester when Dr. BB simultaneously exerted her institutional authority and limited her mathematics expertise authority in ways that prompted students to attribute value to their class contributions, while also considering legitimate practices of the mathematics discipline. It was not until later in the semester that we were able to identify instances where the instructor's *use* of mathematics expertise authority could be perceived as empowering (Research Question 2). We now highlight such a case.

Leveraging Mathematics Expertise Authority

It was about one month into the semester and the Blue Team had just finished their whole-class presentation of a direct proof for the claim, "If l and m are odd integers, then $l+m$ is even." After their presentation, Jayden (a member of the team) suggested, "Also, we could prove it by contrapositive: By showing if $l+m$ is odd, then l and m are even." We believe that Jayden demonstrated agency by offering an alternative proof method unprovoked. He also offered a specific disciplinary technique (contrapositive) that he saw as part of his growing knowledge base. Note however that while Jayden offered a viable alternative proof approach (i.e. contrapositive), his structuring of the contrapositive statement was incorrect. Dr. BB took this as an opportunity to highlight Jayden's suggestion for an alternative proof approach, but to also ensure that the students in the class had an understanding of the correct form for this particular contrapositive statement.

Dr. BB: Let's take one minute, if we were to prove this by contrapositive, what would we need to prove, what would that be?
Natalie: If $l+m$ is odd, then l and m are even integers.
Sofia: I think it is 'l *or* m is even,' because the negation of *and* is *or*.
Dr. BB: The negation of an *and* statement will end up being *or*. So essentially this is like 'l is odd and m is odd.' Then taking the negation of that, this is really important, we need 'l is even or m is even.' Which law is that?
Students: DeMorgan's law.
Dr. BB: Yes. So we can use contrapositive, but make sure we can negate this piece correctly. Okay take a minute to talk in your groups.

In this exchange we see Dr. BB exerting her mathematics expertise authority in two ways. First, Dr. BB recognized Jayden's mistake in the statement of the contrapositive and decided to focus the class's attention on the formation of that statement by pausing the group presentation and asking a probing question. Second, Dr. BB reiterated Sofia's point, summarizing the correct approach to forming the contrapositive by negating the component statements, alluding to DeMorgan's law. Student agency is valued, because of the student-centered nature of the ideas in the discussion and the ability of students to negotiate the correct format of the contrapositive. However, the instructor also honors the discipline by explicitly confirming the correct approach and then having students pause to reason about the proof by contrapositive format as applied to a conjunction statement. In this instance of Dr. BB's exertion of mathematics expertise authority, we see an empowering balance between student agency and students' growing understanding of the norms and truths of the discipline of mathematics.

Discussion and Conclusions

In the results above, we presented brief vignettes of instructor/student interactions that we claim to illustrate the following two situations: (1) An instructor's use of institutional authority that empowered students as legitimate practitioners of mathematics (i.e., Dr. BB's use of institutional authority to set norms within the classroom community that student contributions would be valued and to engage students in an activity that offered them opportunities to consider legitimate disciplinary practices regarding proof), and (2) An instructor's use of mathematics expertise authority that empowered students as legitimate practitioners of mathematics (i.e., Dr. BB's use of mathematics expertise authority to identify a student's mathematical error and to probe students in the class to explore and explain the mathematical error).

These vignettes highlight an instructor's use of authority that contradicts the suggestion by Gerson and Bateman (2010) that "an ideal instructional environment to promote shared authority would limit the instructor's institutional and mathematics expertise authorities" (p. 206). Instead, we offer a more nuanced view of an instructor's use of authority in the teaching and learning of proof, where institutional and mathematics expertise authorities may be used to empower students as legitimate practitioners of mathematics. Further exploration of how different authority types may be used in empowering or disempowering ways would benefit the field.

As we conducted this analysis of classroom transcripts, we noted some interesting connections between institutional authority and mathematics expertise authority. First, as discussed above, we noticed that when the instructor exerted her *institutional authority* in an attempt to set norms that valued student agency, she often simultaneously limited her *mathematics expertise authority*. This may be what Gerson and Bateman (2010) meant in their suggestion to limit mathematics expertise authority, as a way to provide space for students to assert their own mathematical ideas rather than adhere only to the mathematical ideas of the instructor. This led us to wonder, is it possible to exert mathematics expertise authority and simultaneously limit institutional authority? And if so, what would this look like? Institutional authority is an authority type that is at play in the classroom no matter how the instructor decides to act (Amit & Fried, 2005). In fact, when an instructor removes him or herself from contributing to a discussion, they are making use of their institutional authority. So, what would it look like to *limit* institutional authority in an empowering way, or is that possible at all?

We also noted that there were instances within the classroom transcripts where, as researchers, we could clearly distinguish between the instructor's exertion of institutional and mathematics expertise authorities. Our clarity in distinguishing these types of authority is likely due to our advanced knowledge of disciplinary norms in mathematics together with our professional knowledge as instructors. However, we hypothesize that it would not have been as straightforward for the students in the class to differentiate between these authority types. After an instructor makes an authoritative statement, the students may be left to wonder whether they should adhere to the instructor's statement because it would be beneficial for their participation in this classroom community (institutional authority) or whether the instructor is speaking on behalf of the broader mathematics community (mathematics expertise authority). We believe that future research could explore if and when it is important for students to discern between these authority types, and how such discernment aids in their empowerment as legitimate practitioners of mathematics.

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