Mathematicians' Metaphors for Describing Mathematical Practice

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In the literature on metaphor, researchers have pointed out the importance of metaphor as a tool for sense-making and have demonstrated the impact of metaphor use on cognition. In mathematics in particular, metaphor has been shown to be a valuable tool for making sense of and reasoning with mathematics. To our knowledge, there has been no research on the metaphors that professors use when communicating the nature of mathematical practice to students in advanced mathematics lectures. In this paper, we present a particular metaphor, Learning Mathematics is a Journey, that we found in a corpus of 11 advanced mathematics lectures. We describe this metaphor we found and offer some speculative analysis regarding the implications of this metaphor.

Keywords: Metaphors, Mathematical Practice, Advanced Mathematics

A primary goal of contemporary mathematics instruction is to engage students in authentic mathematical activity. (e.g., Ball & Bass, 2000; Lampert, 1990; Rasmussen et al., 2005; Schoenfeld, 1992; Sfard, 1998). To achieve this objective, mathematics educators must grapple with the fundamental question of what it is like to do mathematics. The broad purpose of this paper is to shed light on this issue by exploring how mathematicians describe mathematical activity in their own words. In particular, we analyze the metaphors that mathematicians use when teaching advanced courses for university mathematics students.

We focus on the metaphors that mathematicians use in their lectures for two reasons. First, we know that mathematicians say an important goal of mathematics lectures is to help students understand what doing mathematics is like (e.g., Krantz, 2015; Pritchard, 2010; Rodd, 2003) and previous studies have illustrated some of the ways that professors have described doing mathematics to their students (e.g., Artemeva & Fox, 2011; Fukawa-Connelly, 2012). If we want to study how mathematicians describe their craft in a naturalistic setting, mathematics lectures are a suitable place to look. As we will document in this paper, a particularly common way that mathematicians convey what it is like to do mathematics is by describing mathematical activity metaphorically.

Second, students' perceptions of what mathematics is like are shaped significantly by their experiences in their mathematics classes. Psychological research has demonstrated that the metaphors used to describe a topic exert a powerful influence on how individuals think about that topic (e.g., Thibodeau & Boroditsky, 2011, 2013). Consequently, if we want to study students' perceptions of mathematics, it is important to analyze both the messages that mathematicians convey in their lectures and the ways that students interpret those messages. The analysis in this paper contributes to the first goal. By analyzing the metaphors that mathematicians use in their lectures, we will have a greater understanding of how mathematicians inform students about what it is like to do mathematics.

Theoretical Perspective: Metaphors Structuring Thought

Numerous cognitive scientists and linguists have noted that conceptual metaphors are ubiquitous in the ways humans use natural language, with many scholars claiming that the metaphors that we use structures our thoughts. (e.g., Lakoff & Johnson, 2003; Nuñez, Edwards,

& Matos, 1999; Nuñez, 1998; Reddy, 1979; Thibodeau & Boroditsky, 2011, 2013). In this paper, we follow Nuñez (1998) in defining *conceptual metaphors* (hereafter referred to simply as *metaphors*) as "cross domain 'mappings' that project the inferential structure of a *source domain* onto a *target domain"* (p. 87). As a well-known example, consider Lakoff and Johnson's (2003) claim that we metaphorically view Argument as War. In this example, "war" is the source domain of the metaphor, whereas "argument" is the target domain. This can be seen by many examples that commonly occur in our speech, such as "Your claims are *indefensible*," or "He *attacked every weak point* in my argument" (Lakoff & Johnson, 2003; p. 4). Similarly, Lakoff and Johnson contended that we also metaphorically view Argument as a Journey, such as when we say "We have *set out* to prove that bats are birds," or "We have *arrived at* a disturbing conclusion" (Lakoff & Johnson, 2003; p. 90).

Lakoff and Johnson use these illustrations to highlight three points that will be relevant to this paper. First, metaphor usage is common when we discuss an abstract concept such as argumentation. Second, we can use different metaphors to discuss the same concept and these different metaphors highlight different facets of this concept. For instance, the Argument as War metaphor highlights the combative nature of argumentation, in which arguing is an adversarial activity with an attacker, a defender, a victor, a loser, and so on. The Argument as Journey highlights the sequential and rhetorical aspect of argument in which the individual presenting an argument is trying to lead her audience to a desired conclusion. The third point is the metaphors that we use to describe an abstract concept like argumentation structure our thought about this concept and significantly influence our reasoning about this concept. This third point is the most contentious point (c.f., Glucksberg & McClone, 1999) and we elaborate on this point below.

A central claim advanced by George Lakoff and other linguists is that our use of metaphors is not merely a rhetorical flourish on the part of the speaker. When we use a metaphor, we use our knowledge of the source domain in question to make novel inferences about the target domain. We further give primacy to the aspects of the target domain that are highlighted by the metaphor and less attention to the aspects of the target domain that are ignored by the metaphor.

Empirical support for this position is provided by a series of psychological studies conducted by Thibodeau and Boroditsky (2011, 2013). Thibodeau and Boroditsky noted that we frequently use metaphors when we speak of crime; we sometimes speak of Crime as a Beast in which criminals *prey* on victims and police *track* criminals, *hunt* them down, and *catch* them. We also sometimes describe Crime as a Virus where crime is an *epidemic* that can *plague* a city or *infect* a community. In a series of randomized controlled experiments, Thibodeau and Boroditsky (2011, 2013) compared the responses of participants exposed to different metaphors for crime when the participants were asked to propose measures to reduce crime. Participants who saw the metaphor that crime was a beast uniformly proposed developing better measures to capture and punish criminals. Participants who saw the metaphor that crime was a virus focused more on understanding the social causes of crime and educating the community on how to prevent crime. Thibodeau and Boroditsky concluded that metaphorical usage has "real consequences for how people reason about complex social problems like crime" (p. 1).

The key point to draw from this for the purposes of this paper is that we should not suppose the metaphors that mathematics professors use in their lectures are inert. They are not merely fancy ways of talking, but can say a lot about how mathematicians view their discipline. Further, Thibodeau and Boroditsky's (2011, 2013) studies suggest that these metaphors may influence how students subsequently engage in advanced mathematics.

Existing Literature on Mathematical Metaphors

Research on metaphor usage in mathematics can be divided into two broad areas of studies. First, some scholars have examined how mathematicians and students use metaphors to understand *mathematical concepts*. For instance, Lakoff and Nuñez have explored the use of metaphors in mathematical language in an attempt to understand mathematicians' cognitive underpinnings behind advanced and abstract mathematical ideas (e.g., Lakoff & Nuñez, 2000; Nuñez, Edwards, & Matos, 1999). As an example, mathematicians commonly use the preposition "in" to denote set membership. Lakoff and Nuñez (2000) argued that this use of language suggests that mathematicians metaphorically view sets as containers that are filled with objects. Sfard (1994) and Sinclair and Tabaghi's (2010) interview studies with mathematicians provide empirical support for Lakoff and Nuñez's (2000) theoretical claims. Mathematicians use metaphors as a powerful tool for doing, understanding, and communicating mathematical ideas.

Other researchers have examined how students understand various mathematical topics and concepts through metaphors (e.g. Oehrtman, 2009; Presmeg, 1992; Zandieh, Ellis, & Rasmussen, 2017). For example in interviews with ten undergraduate linear algebra students, Zandieh, Ellis, and Rasmussen (2017) found metaphors to be critical for understanding the varied ways students think about the function concept across high school and linear algebra courses. One student, for example, described one-to-one functions as functions in which, "for every output, there is one input *to get there*" (Zandieh et al., 2017; p. 35). Zandieh et al. identified the language *to get there* as being indicative of a travel metaphor for functions. Zandieh et al.'s (2017) work illustrates that metaphor usage can be used as a lens for studying students' cognition (e.g., the metaphors they use highlight a conception that they possess or are applying) and reveals that metaphors can provide an explanatory account for how students can develop a rich understanding of a concept (e.g., Zandieh and colleagues illustrate how blending metaphors enabled students to unify different conceptions of linear algebra concepts).

A second group of studies has explored metaphors as a lens to understand individuals' beliefs *about* mathematics (e.g. Latterell & Wilson, 2016; Noyes, 2006; Schinck, Neale, & Pugalee, 2008). To date, these studies have focused on students' and primary and secondary teachers' beliefs about mathematics. For instance, Latterell and Wilson (2016) asked prospective teachers to supply metaphors for mathematics. The authors then used these metaphors to understand preservice teachers' attitudes about mathematics. In one example, a student provided the metaphor, "Mathematics is like a tornado in Kansas" (Latterell & Wilson, 2016; p. 287). Latterell and Wilson (2016) suggested this reveals a view of mathematics as something that could cause risk, injury, or harm. Our current paper complements these studies by using metaphor as a lens to investigate mathematicians' beliefs about mathematical activity and how these beliefs may be communicated to their students.

Methods

Participants

In this study, we analyze the metaphors used by mathematicians when giving a lecture in an advanced mathematics course (i.e., a proof-oriented mathematics course for third or fourth year university mathematics students). We recruited participants by sending e-mails to every lecturer at three doctoral-granting institutions in the eastern United States who was teaching an advanced mathematics course. We asked to observe and audio record one of their lectures. Lecturers were not told the purpose of the study. Eleven lecturers agreed.

The Lectures

Each lecture was approximately 80 minutes in length. All professors gave "chalk talk" lectures (Artemeva & Fox, 2011) in which they presented formal mathematics (specifically definitions, theorems, proofs, and examples) on the blackboard. Each class had between seven and 30 students enrolled, with a mean of approximately 18 students. A member of the research team attended and audiotaped each lecture while transcribing everything the lecturer wrote on the blackboard. Each audio recording was transcribed. This transcription was the primary corpus of data used in our analysis.

Analysis.

We analyzed the data following Chi's (1997) scheme for quantifying qualitative analysis of verbal data, which Chi described as a practical guide for making sense of "messy" verbal data. In our first stage of analyses, the first two authors independently read the transcripts flagging for each instance in which a lecturer used a metaphor[1]. Any disagreements were resolved by conversation with all three authors of the paper. 1077 metaphors were identified across the 11 lectures. Of these 1077 metaphors, we found 216 pertained to the activity of doing mathematics.

The second stage of the analysis was thematic; we used an open coding scheme to generate common metaphorical archetypes with a common source domain and a mathematical activity as a target domain. Six metaphorical archetypes having a mathematical activity as a target domain were identified. In the third stage of analysis, we developed clear criteria for what types of utterances counted as an instance of each metaphorical archetype. In the fourth stage, the first two coders independently went through each metaphor that we had previously identified and evaluated if the metaphor belonged to any of the six metaphorical archetypes. Again, any disagreements were resolved through discussion with all three authors of this paper.

The third stage of the analysis was similar to the first. For each of the six metaphorical archetypes, we used thematic analysis to identify particular mappings between a component the source domain and a component of the target domain. For instance, with learning mathematics as a journey, there was often a particular mapping between progress on a journey and formal mathematics covered in a particular class. (e.g., a mathematics professor may say she wants to reach a certain theorem by the end of class, but "we'll see how far I get today"). We would identify criteria by which an utterance could be coded as an instance of this mapping. Then we would go through each metaphor in the metaphorical archetype and evaluate whether each individual metaphor was an instance of that mapping, going back to the original transcript for contextual details if necessary. Again, disagreements were resolved through discussion. The result of following Chi's (1997) methodology is that we can provide an in-depth analysis of the most interesting metaphors that individual professors used but also identify trends across our data set and describe how common these trends were.

As an important theoretical point, the coding scheme that we used is clearly highly interpretive. Following work of Reddy (1979), Lakoff and Johnson (2003), Lakoff and Nuñez (2000), and others, the meanings that we ascribed to the metaphorical utterances exist in the minds of the researchers; our research claim is that other mathematically knowledgeable

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[1] Coding for metaphor usage *in general* was theoretically difficult because metaphors pervade mathematical vocabulary (e.g., Lakoff & Nuñes, 2000). (For instance, is every instance of a professor using the word "in" to denote set membership a use of a metaphor?) However, in this paper, we only discuss metaphors for *mathematical activities* which did not introduce these theoretical nuances. Hence, for the For the sake of brevity, we do not discuss how we resolved disagreements that did not have mathematical activities as a target domain in this paper.

individuals would agree that our interpretations fit well with the data. We cannot be certain if the lecturers themselves intended to convey the meanings that we ascribed to their metaphorical utterances or if students would interpret the metaphorical utterances as we did. We elaborate on this point toward the end of the paper.

Results

Our analysis of these 216 uses of metaphors identified four metaphors used by the eleven mathematicians in the study: Learning Mathematics is a Journey, Doing Mathematics is Work, Mathematics is Discovery, and Mathematics is a Story. Table 2 shows the number of instances of each metaphor in each lecture and the total numbers of instances of each metaphor in the corpus of lectures. As Table 1 reveals, each metaphor was used by at least seven of the 11 lectures we analyzed. For space reasons, we will only provide commentary on the Learning Mathematics is a Journey metaphor here.

Table 1. Counts of Each Metaphor for Mathematical Practice by Lecture

Metaphors	Instances in Each Lecture											Total Instances
	L1	L2	L3	L4	L5	L6	L7	L8	L9	L10	L11	
Journey	1	1	2	2	0	3	2	6	14	0	8	39
Work	10	5	2	4	3	10	7	4	12	1	1	59
Discovery	7	1	3	22	5	5	0	0	10	25	13	91
Story	5	0	2	0	1	0	1	8	7	0	3	27

Learning Mathematics is a Journey

Table 2. Metaphor Map for the Learning Mathematics is a Journey

Source Domain: Journey	Target Domain: Mathematics	Number of Instances**	Example
Progression along the journey	Progression in one's learning of mathematical content	16 (6)	L8 [abstract algebra]: "I'd like to do at least a little bit on group theory, *we may or may not get to that*."
Times/Locations along a Journey	Particular mathematical content learned at a particular time in one's mathematical career	16 (3)	L1 [set theory]: "I'd like to get to today, or very soon, is that this notion of cardinal arithmetic *will allow you to get away from the very explicit arguments that we've been doing the last few weeks*."
Journeys can have required landmarks or checkpoints to be crossed	Important Mathematical Ideas and Concepts that should be learned in mathematics	5 (5)	L2 [real analysis]: "So I think that you *can't possibly have gotten very far in math here without having seen Euler's number e*, which is usually defined this way."
Other		6 (4)	L3 [number theory]: "The proof of this will wait a little bit. Okay? We won't

** *Number of instances across the lectures (Number of lectures in which the instances occurred).*

On a long journey, travelers depart from an initial location and set out with a particular destination in mind. The traveler's journey may span multiple days in which they plan to traverse a certain distance and reach a certain point at the end of a day. In the lectures, mathematicians spoke of particular mathematical topics or results in terms destinations that they hoped to reach. The mathematical topics that were covered were analogous to the ground that could be covered on a particular leg of a journey.

The metaphor of Learning Mathematics is a Journey was often invoked at the start of the lecture; six lecturers initiated their lectures by using this metaphor to describe the planned itinerary for the day and the metaphorical location that they hoped to reach by the end of the lecture. For example, L8 used three metaphors of this type in the first six minutes of her lecture: “I’d like to do at least a little bit on group theory, *we may or may not get to that*”, “at least *if we get through this* chapter 6, it’ll be a nice ending for you *if we don’t get further*”, and “so *we’ll see how far I get* today.” The common theme in these quotations is that L8 wanted to cover certain topics (a little bit of group theory, the end of chapter 6) that he metaphorically described as locations that he would like the class to reach by the end of the lecture.

The discussion of L8 above used the journey metaphor in a local sense in describing what ground would be covered in a particular lecture. However, the lecturers sometimes used journey in terms of students’ mathematical development, either in terms of the entire semester or even beyond that (see L2’s quote in Table 2). In a journey, there may be particular landmarks that a traveler wishes to see or has seen in the past. The lecturers would describe important mathematical results as being these landmarks. For instance, in the last fifteen minutes of the lecture, L9 talked about the distance from and progress made toward a mathematical accomplishment in terms of distance from and progress toward a physical destination. After discussing how the real numbers are an extension field of the rational numbers, L9 said “here, *we are on the verge* of synthesizing or generalizing that approach” where the “verge” is “the edge or border of something” (Cambridge English Dictionary). The class was approaching an accomplishment of defining an extension field given an arbitrary field. In L9’s language indicating they were *on the verge* suggests to us that the class was approaching a desired mathematical destination.

Meanwhile, we see that the class still had some ground to cover before arriving at this destination. L9 said, “now that’s promising in that this sets this up as a direct parallel to this, but it doesn’t yet, on its own, guarantee that *we have gone far enough* to find a root for P(x), okay? […] Now, *we won’t reach that pinnacle* today unfortunately.” In this quote, L9 explained that the class has not yet *gone far enough*, or made sufficient progress, in their journey to reach the desired destination meaning that they have not covered the necessary content required to complete this generalization. As such, we see L9 describing the class’s current or local progress on the journey in relation to the broader journey of learning mathematics. In particular, L9 also described the destination as a *pinnacle*, suggesting that this destination is not simply the next stop on the journey, but rather a local maximum in the domain being covered and a critical landmark that the students want to reach. Next, L9 paused to allow students to ask questions and continued “the *farther we go in this*, the more focused I can be in my anticipations of it.” Here, L9’s language suggests that as the class continues to make progress toward their anticipated mathematical destination, he will be able to better anticipate questions and guide the class on their journey.

Discussion

The main results of this study are that mathematics lecturers regularly invoke metaphors when they describe the activity of doing mathematics. Further, there was overlap in the metaphors that were used by different mathematicians. We first offer speculative thoughts about the implications of the metaphors we found, Learning Mathematics is a Journey, Doing Mathematics is Work, Mathematics is Discovery, and Presenting Mathematics is a Story. We then conclude the paper by delineating the limitations of our study and suggest directions for future research.

In undergraduate mathematics education, mathematicians frequently express an obligation to cover a certain amount of mathematical content. In contrast, most mathematics educators believe that content coverage by itself is useless if students do not understand the material that is covered (e.g,. Fukawa-Connelly et al., 2017). In the Learning Mathematics as a Journey metaphor, the lecturers frequently spoke of the ground they needed to cover and the landmarks they intended to reach. However, there were only one instance in which a lecturer (L3) mentioned losing students on the journey. One possibility is that lecturers' use of the Learning as a Journey metaphor provides a lens into their obligations as a teacher (covering ground and reaching destinations) and what are peripheral considerations (the number of students who are able to actually complete the journey). This also suggests how a change in the metaphor might lead some mathematicians to reconsider what they value. After all, a Sherpa who successfully scales Mount Everest would not be regarded as successful if the majority of his party perished along the way. Adding the notion of "survival rate" to the Learning Mathematics as a Journey metaphor could add the aspect of student learning to the metaphors that mathematicians used.

In terms of limitations, we emphasize that we only looked at mathematicians' metaphor usage in the context of lectures in United States classrooms. In light of recent inquiries into how language and culture shape mathematical curricula (Shinno et al., 2018), it would be worthwhile to investigate whether mathematicians from other cultures than our own used different metaphors to describe mathematical practice. The theoretical work of Lakoff and Johnson (2003) and the empirical work of Thibodeau and Boroditsky (2011, 2013) demonstrate how the metaphors that are used to frame a concept influence how people reason about that concept. It would be interesting to investigate how, if at all, the particular metaphors that mathematicians use influence students' mathematical reasoning and epistemologies. Questions regarding the impact of metaphors on student thinking and reasoning may have the potential to generate very interesting theoretical and empirical research.

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