How do Transition to Proof Textbooks Relate Logic, Proof Techniques, and Sets?

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Many mathematics departments have transition to proof (TTP) courses, which prepare undergraduate students for proof-oriented mathematics. Here we discuss how common TTP textbooks treat three topics ubiquitous to such courses: logic, proof techniques and sets. We show that these texts tend to overlook the rich connections sets have to proof techniques and logic. Recent research has shown that student thinking about sets is propitious to novice students' ability to reason about logic and construct valid arguments. We suggest several key connections TTP courses can leverage to better take advantage of their unit(s) on sets.

Keywords: transition to proof, textbook analysis, logic

Introduction

Over the past few decades, many mathematics departments have recognized the need to help students through two major undergraduate transitions: the transition to college mathematics and the transition to proof-oriented mathematics. A recent survey found that the majority of mathematics departments at research universities have attempted to address the latter by creating 'transition to proof' courses specifically aimed at helping students navigate the challenges of proof-oriented mathematics (David & Zazkis, 2017). The content of such transition to proof (TTP) courses can be quite diverse, but they often include a number of topics that are necessary – though perhaps not sufficient – for learning how to read and write proofs in later courses. Specifically, such courses usually address mathematical logic, sets, and basic proof techniques. We consider these topics necessary but not sufficient because understanding them will not guarantee success in later courses, but violating logical laws, misusing set structures, or using invalid proof techniques will almost certainly undermine later success. Mathematical logic, sets, and basic proof techniques are ubiquitous amongst transition to proof courses (David & Zazkis, 2017), and thus we expect and proof-oriented course to draw upon ideas from each of these domains. Each of these topics also corresponds to an entire field of mathematics – formal logic, set theory, proof theory – so that any one topic could fill an entire course. Instructors of such courses must therefore make careful pedagogical choices about what and how much to introduce from each of these domains.

Little is known, however, about the results of these choices – that is, how logic, sets, and proof techniques are presented in transition to proof courses. To gain insight into this issue, we analyzed how these three topics are covered and connected in commonly used transition to proof (TTP) textbooks. Our inquiry was guided by the following research questions: How are basic ideas of logic, sets, and proof techniques introduced and explained? How do TTP books connect these domains? In what order do they appear?

Literature and Theoretical Perspective

In this section we consult relevant literature on student thinking about sets, logic, and proof techniques in order to present the beginnings of a *conceptual analysis* (Thompson, 2008), a theoretical model that describes "ways of knowing that might be propitious for students' mathematical learning" (p. 46). We operationalized our conceptual analysis as a lens through which to investigate and compare the presentation of these topics amongst our textbook sample.

We chose to focus on these three topics because, in addition to their ubiquity in TTP courses (David & Zazkis, 2017), they each provide some necessary contribution to understanding proof-oriented mathematics. Furthermore, there are common elements of mathematical text that simultaneously draw upon all three topics.

Recent studies on student thinking about logic (Dawkins, 2017; Dawkins & Cook, 2017; Hub & Dawkins, 2018) have investigated how students read mathematical statements prior to being taught formal logic; the students' intuitive approaches and interpretations in these studies were compared to the normative ways of interpreting such language. One of the key findings of this series of studies was that students who connected mathematical categories (e.g. "rectangle," "even," "divisible by 4") to the sets of objects in the category were able to adopt expert ways of reading mathematical language much faster than their peers (who focused on examples or properties). They were also better at forming valid arguments for why quantified statements were true. Moreover, building the truth table for logical connectives was insufficient for students to successfully build strategies that mirrored Venn diagrams unless they were conversant in thinking about sets. In other words, adding quantifiers posed a significant challenge to students' ability to verify and falsify statements and to formalize their ideas about logic, even when they understood the truth table for a connective. Based upon these studies, we contend that being able to relate set ideas to logic and proof techniques is key – that is, thinking about sets is propitious to novice students' ability to reason about logic and construct valid arguments.

Consider the following example of how the ability to move flexibly amongst understandings that center on logic, sets, and proof techniques might afford different insights in the context of interpreting the following conditional statement (which, conceptually, amounts to stating that divisibility is a transitive relation): "Let $a, b,$ and c be integers. If $a|b$ and $b|c$, then $a|c$."

1. Logically, we might assert that the theorem is true because we cannot find three numbers such that $a|b,\ b|c$, and $a \nmid c$. In other words there does not exist a case that makes the antecedent true and the consequent false.
2. Set-wise, conditionals always connect to subset relations. In this case, the theorem can be restated as $\{(a, b, c) \in \mathbb{Z}^3 : a|b \text{ and } b|c\} \subseteq \{(a, b, c) \in \mathbb{Z}^3 : a|c\}$. If we pick any triplet in the first set, we know that it will necessarily be in the second set.
3. In terms of proof techniques, we might say that the property $a|c$ can be inferred from the properties $a|b$ and $b|c$. Alternatively, $a \nmid b$ or $b \nmid c$ might be provable from $a \nmid c$. Lastly, it may be that $a|b$, $b|c$, and $a \nmid c$ are inconsistent.

While these may seem like subtle distinctions, they have the potential to provide potentially valuable information. The first view discusses truth-values or what kinds of triplets of integers exist. The second view emphasizes how the predicates in the theorem range over all of $\mathbb{Z}^3$ and each have a truth-set they represent. The relationship between the antecedent and consequent properties can be understood as a relationship between these truth-sets (Hub & Dawkins, 2018). The third view draws our attention to the inferences available from the hypotheses of the theorem, such as creating equations $a = mb$ and $b = nc$ for some $m, n \in \mathbb{Z}$ and using substitution to proceed with the proof. Stated this way, it seems that the first and last interpretation are most mathematically useful for the work of reading and writing proofs. However, we posit that students should understand – and, as a consequence, TTP textbooks should address – logic and sets because it appears fruitless to be able to write a valid proof if one does not understand the second and third interpretations as entailments of that proof.

Methods

Our objective was to obtain a sample of TTP textbooks that accurately reflect those in widespread use in undergraduate classrooms in the United States. To do so, we leveraged the results of a recent study of TTP courses (David & Zazkis, 2017), which analyzed the syllabi from TTP courses at all institutions categorized by Carnegie designations as high research activity and very high research activity in the United States. The study reported which portion of those courses used a textbook and which textbooks were most commonly used. To ensure that our sample was reasonably representative yet still tractable enough to allow for detailed individual analyses, we selected those textbooks in use at a minimum of 6 universities (as reported by David & Zazkis, 2017). We included one more book intended for inquiry-based TTP instruction in order to guarantee our sample was more diverse in terms of instructional approaches. A complete bibliography of the textbooks in our sample is included after the references.

After obtaining copies of all 10 textbooks in our sample, the data collection process initiated with each researcher independently reading the front matter (e.g. preface, notes to the instructor and/or student) of a particular text to gain insight into any global themes and general strategies for content presentation. Notes were recorded about any approaches that seemed to place strong emphasis on one of our three main topics (logic, sets, and proof techniques). Next, each researcher used the table of contents and the index to identify the places in each text where logic, sets, and proof techniques appeared. We recorded excerpts and quotations that we deemed provided insight into connections between logic, sets, and proof techniques – as described in our conceptual analysis in the previous section – in a spreadsheet. Each textbook was reviewed by at least two members of the research team. We used *constant comparison* (Creswell, 2007; 2008) of textbook materials to identify common themes across the data set, including common sequences in which logic, proof techniques, and sets appeared in each text and how that might have influenced their presentation of each.

Results, part I: Overview of Textbook Sample

Four general points emerged from a global analysis of our entire sample[1]. First, sets appeared to be the one element that varied in position most widely across the texts. Collectively, logic (L), quantification[2] (Q), and proof techniques (P) most often appeared in the order L – Q – P (seven texts) or Q – L – P (two texts). Sets almost evenly varied between appearing first (four texts), in the middle (three texts), and last (three texts).

Second, the most common connection that textbooks made among the logic, proof techniques, and sets was to explain or justify proof techniques using truth tables. We will consider some examples of these explanations in a later section.

Third, about half of the texts connected logic and sets in explicit ways. Four textbooks explained set ideas using logical structure. This seems a natural approach since one can translate set operations – $A \cup B$ – into set-membership conditions with logical connectives – $\{x: x \in A \; or \; x \in B\}$. Alternatively, a natural way to introduce the notion of set itself is through truth-sets for predicates (e.g. the set of multiples of 4, the set of divisors of 52). While some books used

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[1] It is beyond the scope of this brief proposal to present the results of our analysis of each textbook in our sample. Those who are interested to see such details may follow this link (https://www.dropbox.com/s/jhrxo8bajhzon8s/Rume2019proposal-table.pdf?dl=0) to a table that includes (1) the authors and names of the books we analyzed, (2) the order in which logic, proof techniques, quantification, and sets appeared, and a brief summary of how each textbook connected the topics in question.

[2] Quantification is often understood as part of logic, but we found it useful to distinguish it because some books dedicated long sections to only propositional logic (without quantifiers) and others dedicated more time to predicate logic (quantified). Also, quantifiers themselves varied from being treated as logical constants to being phrases in mathematical language. In other words, quantifiers sometimes were treated more logically (in terms of truth-conditions) and other times more linguistically (what do these phrases mean and how do we use them).

this as an introduction (i.e. the first explicit mention of sets), they often shifted to talking about sets as general collections without some underlying predicate. Only one book explicitly built the truth-table for a (quantified) conditional statement by considering the truth of an example statement on various sets of inputs. In this case, the set structure guided the exposition of logic.

Fourth, sets generally played no role in explaining or justifying proof techniques. Rather, the primary examples of connections between proof techniques and sets occurred when sets were discussed last and thus the other topics informed the exposition of sets.

Results, part II: Analysis of Illustrative Excerpts from Textbook Sample

The above summary of our textbook analysis findings suggests that TTP textbooks frequently link logic and proof techniques and with some regularity connect sets to logic. Sets in particular appear the most isolated of the three topics. This forms a simple descriptive account of current TTP curricula. We pursue two goals hereafter. First, we will provide some excerpts from the textbooks that illustrate the nature of the connections between logic, sets, and proof techniques to recognize some qualitative differences that likely matter for student sense making. We shall also note some potential connections that, according to our conceptual analysis, could have been made that were not, specifically with regard to sets.

As stated above, the most common connection TTP books made among logic, proof techniques, and sets was to motivate proof techniques for conditional statements by demonstrating their validity through the use of truth tables. Below we provide some excerpts from the books that illustrate how this was done. Overall, we notice that the books draw upon diverse resources to help students make sense of proof techniques.

Solution: Let p be the proposition "You send me an e-mail message," q the proposition "I will finish writing the program," r the proposition "I will go to sleep early," and s the proposition "I will wake up feeling refreshed." Then the premises are $p \to q$, $\neg p \to r$, and $r \to s$. The desired conclusion is $\neg q \to s$. We need to give a valid argument with premises $p \to q$, $\neg p \to r$, and $r \to s$ and conclusion $\neg q \to s$.

This argument form shows that the premises lead to the desired conclusion.

Step	**Reason**
1. $p \to q$	Premise
2. $\neg q \to \neg p$	Contrapositive of (1)
3. $\neg p \to r$	Premise
4. $\neg q \to r$	Hypothetical syllogism using (2) and (3)
5. $r \to s$	Premise
6. $\neg q \to s$	Hypothetical syllogism using (4) and (5)

Figure 1. Rosen's (2012, p. 74) example proof connecting proof techniques to rules of inference.

A **direct proof** of a conditional statement $p \to q$ is constructed when the first step is the assumption that p is true; subsequent steps are constructed using rules of inference, with the final step showing that q must also be true. A direct proof shows that a conditional statement $p \to q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs. In a direct proof, we assume that p is true and use axioms, definitions, and previously proven theorems, together with rules of inference, to show that q must also be true. You will find that direct proofs of many results are quite straightforward, with a

Figure 2. Rosen's (2012, p. 82) explanation of direct proof techniques using truth tables.

From an early stage, Rosen (2012, Fig 1) invites students to cite rules of inference (e.g. "contrapositive" and "hypothetical syllogism") as warrants in proofs. The example theorem does not concern mathematics and the author immediately replaces the propositions with logical variables to construct a proof in logical syntax. The text's later examples are mathematical and quantified and Rosen uses predicates to explain proof by universal generalization. However, when the author explains the proof technique (as shown in Fig 2), the language shifts back to propositional variables and "assumption" of the hypothesis rather than selecting an arbitrary element of the truth set of the hypothesis predicate.

The table shows that if P is false, the statement $P \Rightarrow Q$ is automatically true. This means that if we are concerned with showing $P \Rightarrow Q$ is true, we don't have to worry about the situations where P is false (as in the last two lines of the table) because the statement $P \Rightarrow Q$ will be automatically true in those cases. But we must be very careful about the situations where P is true (as in the first two lines of the table). We must show that the condition of P being true forces Q to be true also, for that means the second line of the table cannot happen.

This gives a fundamental outline for proving statements of the form $P \Rightarrow Q$. Begin by assuming that P is true (remember, we don't need to worry about P being false) and show this forces Q to be true. We summarize this as follows.

Outline for Direct Proof

Proposition If P, then Q.

Proof. Suppose P.

$\vdots$

Therefore Q. ■

Figure 3. Hammack's (2013, p. 92) explains direct proof of conditionals using the truth table.

Hammack's (2013, Fig 3) representations, which closely mirrored several others, present general proof frames using propositional variables, though mathematical example proofs appeared nearby for comparison. He explains the initial step "Suppose *P*" in light of the fact that $P \Rightarrow Q$ is always true when P is false (similar to Rosen). Interestingly, the examples all involved predicates, but Hammack presents the proof techniques using only the proof table and propositional variable.

2.21. Remark. *Logical connectives and membership in sets.* Let $P(x)$ and $Q(x)$ be statements about an element x from a universe U. Often we write a conditional statement $(\forall x \in U)(P(x) \Rightarrow Q(x))$ as $P(x) \Rightarrow Q(x)$ or simply $P \Rightarrow Q$ with an implicit universal quantifier.

The hypothesis $P(x)$ can be interpreted as a universal quantifier in another way. With $A = \{x \in U :\ P(x) \text{ is true}\}$, the statement $P(x) \Rightarrow Q(x)$ can be written as $(\forall x \in A)Q(x)$.

Another interpretation of $P(x) \Rightarrow Q(x)$ uses set inclusion. With $B = \{x \in U :\ Q(x) \text{ is true}\}$, the conditional statement has the same meaning as the statement $A \subseteq B$. The converse statement $Q(x) \Rightarrow P(x)$ is equivalent to $B \subseteq A$; thus the biconditional $P \Leftrightarrow Q$ is equivalent to $A = B$.

2.24. Remark. *Elementary methods of proving $P \Rightarrow Q$.* The *direct method* of proving $P \Rightarrow Q$ is to assume that P is true and then to apply mathematical reasoning to deduce that Q is true. When P is "$x \in A$" and Q is "$Q(x)$", the direct method considers an *arbitrary* $x \in A$ and deduces $Q(x)$. This must not be confused with the invalid "proof by example". The proof must apply to every member of A as a possible instance of x, because "$(x \in A) \Rightarrow Q(x)$" is a universally quantified statement.

Remark 2.20f suggests another method. The **contrapositive** of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$. The equivalence between a conditional and its contrapositive allows us to prove $P \Rightarrow Q$ by proving $\neg Q \Rightarrow \neg P$. This is the *contrapositive method*.

Remark 2.20c suggests another method. Negating both sides $(P \Rightarrow Q) \Leftrightarrow \neg[P \wedge (\neg Q)]$. Hence we can prove $P \Rightarrow Q$ by proving that P and $\neg Q$ cannot both be true. We do this by obtaining a contradiction after assuming both P and $\neg Q$. This is the *method of contradiction* or **indirect proof**. We summarize these methods below:

Figure 4. D'Angelo and West's (2000, pp. 34,35) explanation of conditional proof methods with reference to quantification.

D'Angelo & West (2000, Fig 4) directly address how proofs of conditionals verify quantified claims making use of the connections they previously established between sets and logical relations. The explanation uses logical variables, though the authors immediately provide mathematical examples thereafter. D'Angelo and West's explanation seems to provide the most attention to the sets underlying the predicates while still using logical variables for exposition.

Thinking about an example should help. Consider the statement "If x is an even integer, then x^2 is an even integer." I suspect that when you conducted the "thought experiment" you decided that this is true. It is a case in which there are infinitely many values of x that make the hypothesis true. So we will have to assume (in the abstract) that x is even and then show that x^2 has to be even, too.

If we are to get anywhere, we first have to recall what it means to say that an integer is even:

When an implication is proved by assuming that the hypothesis is true and then showing that the conclusion is also, the proof is called a **direct proof**.

Here is the proof that if x is an even integer, then x^2 is an even integer.

Proof. Suppose that x is an even integer. Then by definition of even integer, we know that there must exist an integer y such that $x = 2y$. Now we have to show that there is an integer w so that $x^2 = 2w$. Let $w = 2y^2$. Since the product of integers is an integer, $w = 2y^2$ is an integer. Notice that

$$x^2 = (2y)(2y) = 2(2y^2) = 2w.$$

Thus x^2 is an even integer. ■

This argument works for *any* even number; thus all cases have, in some sense, been checked.

Figure 5. Schumacher's (2001, p. 32-33) explanation of direct proof of a quantified conditional.

Schumacher's (2001, Fig 5) presentation attends more directly to quantification, though the quantifiers themselves stay implicit throughout. Her example theorem is mathematical and she does not rely on logical variables to present the proof.[3] She points out that the hypothesis of the theorem is true for infinitely many values of x, so the proof must work for all such values. Woven throughout the exposition is the assumption that "assuming that the hypothesis is true" is tantamount to selecting (any) even value of x.

Discussion

To summarize, the presentations of proof techniques vary from constructing derivations within a propositional logical calculus (in which every step is validated by a rule of inference) to mathematical proofs (in which familiar mathematical content is written in paragraph format using warrants that would likely be familiar to TTP students). Many of the presentations exist between these poles of operating in a logical calculus and examining actual mathematical proofs. Many books explain patterns or strategies in proof construction using logical variables with varying levels of attention to the quantification structure that is present in most of the mathematical proofs constructed later in each text. We offer two primary observations about how these common intermediate approaches may be problematic for students.

First, these textbooks tend to use propositional variables to explain proof techniques that are almost always applied to situations involving predicates. We are sensitive to this trend in light of our experiences researching how novice students interpret mathematical language. When many students read a phrase such as "x is an even number," they are frequently drawn to select a representative even number (or to think about properties such as the units digit being even). Many students need guidance to understand the way that mathematicians infer that this phrase almost always implicitly refers to any even number (unless x is already a bound variable). By referring to these phrases in proofs as propositions, we worry that these TTP texts might reinforce this limiting trend in student reasoning. Assigning truth-values ("assume P is true") does not help students attend to the underlying set structure ("select an arbitrary x from the set of even numbers"). Similarly, the suppression of quantification is common in mathematical proof writing. Indeed, there are likely many familiar theorems that we have never thought about using the subset interpretation mentioned by D'Angelo and West (2000; Fig 4). Our contention is that texts that teach students how to read and write proofs (maybe for the first time) might need to give students more time to understand the role of quantification and sets in proof techniques before these ideas can be left implicit. This matter becomes especially challenging for students when we consider falsifying statements by counterexample or negating statements.

Second, representing proof techniques using logical variables may preclude students' ability to make sense of the set structure that underlies common proof techniques. What we mean is that when students read a meaningful mathematical predicate such as "x is even," "$a|b$," or "$2n^2 + 3$ is a multiple of 5," there is at least the opportunity for them to reason about the truth set of the predicate. However, when TTP books explain proof techniques using logical variables such as P, we expect students to find thinking about $\{x \in U: P(x)\}$ to add little insight. In contrast, we concur with Schumacher's (2001) effort to draw students' attention directly to the way that proofs written using definitions apply to all objects that satisfy the definition. This is part of what Dawkins (2017) refers to as *reasoning with predicates*, which refers to students' propensity to associate with any mathematical category the set of objects in the category. In our research, we

[3] Earlier in the text she invited readers to prove logical equivalences or differences using truth tables, noting that the logical variables there stood for predicates.

find that students do this much more easily with familiar categories such as even numbers, multiples of a, or factors of b. This seems reasonable since they have had experience with such sets since grade school and can anticipate how those sets would be populated. Students need some guidance and experience thinking about the truth sets of negatively stated predicates ("f is a non-continous function") and unfamiliar categories ("$2n^2 + 3$ is a multiple of 5"). Once again, we acknowledge that experts may often write proofs without thinking explicitly about these underlying sets. We contend, though, that novices often do not find such connections immediate when they are learning to read proofs; reading valid proofs without such understanding leaves something to be desired.

Conclusion

We close by proposing a few goals for TTP instruction. We prioritized these goals because 1) our research leads us to question whether students will make these connections unless they are explicitly accounted for in instruction, and 2) our textbook analysis herein reveals that sets are the most underdeveloped of the three core topics we examined.

1. Recognizing that every predicate entails an underlying truth set and membership in any set can be understood as a predicate. We anticipate that it might be helpful to build up to this generalized relationship by starting with familiar sets (even numbers), before moving to property-based predicates ($\{n \in \mathbb{Z}: 5|2n^2 + 3\}$), before thinking about generalized predicates ($P(x)$ is true if $x \in \{1,5,7\}$).
2. Recognizing the set over which the predicates in a theorem range. Many theorems involve a number of variable elements that each constitute a variable in the theorem's predicates. Helping students attend to the variables and their scope is an important part of understanding what a theorem says and what a proof accomplishes. Indeed, this seems one of the most natural ways to see the importance of Cartesian products of sets.
3. Connecting the various ways to interpret mathematical texts listed above: the statement "Suppose [$P(x)$ is true]" can be thought of as assuming the hypotheses true, selecting an arbitrary x in the scope of the predicate P, as beginning proof by universal generalization, or as providing the assumptions from which we must deduce the theorem's conclusions. Part of the work of the TPP course is to help students understand why all of these are accomplished by the same text.

Overall, our analysis of Transition to Proof texts revealed that textbooks intended for such courses frequently connect logic and proof techniques, and connect logic and sets. However, they infrequently connect sets to proof techniques. Indeed, analyzing the representations used to introduce proof techniques reveals that it would be hard to make sense of the underlying truth sets because hypotheses are so often represented by logical variables. Our research suggests that students need help thinking about the underlying sets and that this can help them reason about logic and argumentation. Accordingly, we argue that TTP courses should help students connect assumptions of truth with arbitrary selections from particular sets. We offer this reflection to encourage instructors to think about and attend to the potential for such connections in TTP courses. Ultimately, we hope such considerations can help more of our students succeed in learning how to read, write, and truly understand mathematical proving, thereby gaining access to its great epistemic power.

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