How Do Mathematicians Describe Mathematical Maturity?

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The concept of mathematical maturity is one that, for some, elicits clear meanings and perhaps illustrations of ideal mathematical students. Mathematicians have been reported to use this term in various ways, yet there is no clear or empirically based description of mathematical maturity at this time. This proposal explores existing descriptions of mathematical maturity as well as descriptions of the related concepts of mathematical intuition and mathematical beliefs. This proposal reports preliminary findings from interviews with mathematicians investigating their understandings of mathematical maturity. Preliminary results include three main components of mathematical maturity: ways of thinking about mathematics, mathematical intuition, and comfort with and competence in mathematics.

Keywords: mathematical maturity, mathematicians, advanced mathematics courses

"Mathematical maturity" is a term often used by many mathematicians to describe some collection of desirable features in their advanced undergraduate students. In some cases, mathematical maturity can even be listed as a prerequisite requirement for advanced mathematics classes or as a learning objective in undergraduate course descriptions and syllabi. The concept is ubiquitous enough within practice to warrant a Wikipedia page which provides the following description:

> Mathematical maturity is an informal term used by mathematicians to refer to a mixture of mathematical experience and insight that cannot be directly taught. Instead, it comes from repeated exposure to mathematical concepts. It is a gauge of mathematics student's erudition in mathematical structures and methods.

Meanwhile, this description is worrisome for multiple reasons. First, describing mathematical maturity as an informal term suggest that it is not used in an official capacity. Second, describing mathematical maturity as "a mixture of mathematical experience and insight" is remarkably vague. Third, the perspective that mathematical maturity cannot be taught is suggestive of a perspective that a learner can inherently succeed at mathematics or they cannot.

This proposal explores the concept of mathematical maturity from the perspective of mathematicians discussing their understanding of mathematical maturity in the context of their undergraduate students. The literature review and theoretical perspective ground the discussion in published opinions of expert mathematicians, mathematical philosophy, and existing work on related topics. Preliminary results from interviews with five pure mathematicians highlight three main components of mathematical maturity and ways these mathematicians report to foster mathematical maturity in their students.

This study investigates the following research questions: How do mathematicians describe mathematical maturity? Is there a difference between how pure mathematicians and applied mathematicians describe mathematical maturity?

Literature Review and Theoretical Perspective

While mathematical maturity may be viewed as a goal of university mathematics and a characteristic of an ideal advanced undergraduate student, the treatment of mathematical maturity in the literature does not reflect this importance. Steen (1983) described mathematical maturity as impossible to define, but suggested that there are “several marks of maturity that most mathematicians will instantly recognize” (p. 99). These marks include the ability to abstract and the ability to synthesize. Steen continues to identify additional “criteria of maturity” all listed as various abilities (for instance, the abilities to use and interpret mathematical notation, to generalize, to perceive patterns) she believed mathematically mature students must possess. Steen’s essay is based on the author’s and her colleagues’ professional opinions, and very little has been published on the specific topic of mathematical maturity since.

Meanwhile, a number of famous mathematicians have published their opinions around their understanding of mathematical thinking, its development, and the mathematical education of an individual. For instance, Tao (2007) argues that mathematical education can be roughly divided into three stages: a pre-rigorous stage, the rigorous stage, and a post-rigorous stage. He suggested that mathematics is first taught informally, then students are taught to be more precise and formal, before they are ultimately to return to the informal—using their intuition which is now supported by their comfort with the “rigorous foundations”. Whereas, Thurston (1998) suggested that human language, visual/spatial sense, logic/deduction, intuition/association/metaphor, stimulus-response, and process/time are important for mathematical thinking.

These various existing descriptions of mathematical maturity and important characteristics of successful mathematical thinking and learning do not offer empirically tested frameworks of mathematical maturity, but do provide an impression of what might be important or necessary aspects of mathematical maturity. In particular, these descriptions suggest the significance of a learner’s mathematical beliefs and intuitions, which are discussed below.

Mathematical Beliefs

The existing literature on mathematical beliefs is expansive and diverse in its interpretation of the concept. Muis’s (2004) review of 33 studies on students’ epistemological beliefs highlighted this variety and focused on “beliefs about the nature of mathematical knowledge and mathematical learning” (p. 324). In this review, Muis identified that much of the research surrounding students’ mathematical beliefs found the students to possess beliefs that are nonavailing—or those that either do not influence or negative influence learning outcomes. For instance, the literature reviewed identified students as believing that 1) mathematical knowledge is unchanging, 2) the goal in mathematics is to find the correct answer, 3) knowledge is delivered by an authority, 4) mathematical ability is innate, 5) components of mathematical knowledge are unrelated, and 6) students are incapable of constructing knowledge and solving problems on their own. Muis went on to explain the negative effects of students’ nonavailing beliefs on their strategies for learning and motivational orientations, calling for future research considering the impact of teachers’ epistemological beliefs and their instructional styles on students’ beliefs.

Indeed, research has shown that students’ experiences within their mathematics classrooms are highly influential in shaping students’ beliefs about mathematics. For example, Schoenfeld’s (1989) survey of 230 high school students’ mathematical beliefs highlighted the separation in students’ minds between the procedural, rote mathematics they were accustomed to seeing in their schools from the interpretive and creative nature of mathematics.

Moreover, the mathematics education literature has further shown that students’ mathematical beliefs affect their understanding and study of mathematical content. For instance, Szydlik’s (2000) study of 27 calculus students indicated that of the students interviewed, those with “internal sources of conviction provided more static definitions […] and fewer incoherent definitions than students with external sources of conviction” (p. 272). As such, students with nonavailing beliefs, such as believing that the goal of mathematics is to successfully receive knowledge from an authority to directly use the knowledge to achieve the correct answer, may have more difficulties understanding mathematical content than those with availing beliefs.

Mathematical Insight and Intuition

The above descriptions of mathematical maturity given by Wikipedia and Steen (1983), as well as the descriptions of the learning and thinking of mathematics by Tao (2007) and Thurston (1998) all suggest the necessity of mathematical insight and mathematical intuition.

Meanwhile, definitions of both mathematical insight and mathematical intuition seem to have largely evaded the literature. For instance, Keijzer and Terwel (2003) claim that "mathematical insight is widely recognized as an important goal of education", despite failing to provide a definition of the term. Hartmann (1937) offered the generic definition of insight as the "process of making an organism aware of the conditions governing the phenomena to which it is reacting" (p. 19), but does not extend this discussion to explain precisely what is mathematical insight. Griffiths (1971) even suggested that it would be impossible to define mathematical insight and instead offered examples and anecdotes of theoretical students who lacked mathematical insight. Griffiths does continue to suggest that mathematical insight and mathematical intuition are not the same concept; however, others disagree and use them interchangeably.

Mathematical intuition has been more explicitly discussed in the literatures of mathematical philosophy and mathematics education. For instance, Feferman (2000) describes intuition as the "insight or illumination on the road to the solution of a problem" (p. 317) and a "mathematicians' hunches as to what problems it would profitable to attach, what results are expected, and what methods are likely to work" (p. 318). Similarly, Fischbein (1982) described intuition as the unconscious ability to "organize information, to synthesize previously acquired experiences, to select efficient attitudes, to generalize verified reactions, to guess, by extrapolation, beyond the facts at hand" (p. 12). Tall (1980) described intuition as "the global amalgam of local processes from the current cognitive structure selectively stimulated by a novel situation" (p. 2). Thus, we do see an acknowledgement of an unconscious or semi-conscious aspect of intuition as well as the problem-solving aspect of intuition in each of these descriptions above. It is further noteworthy that in Burton's (1999) study involving interviews with 70 mathematicians, most of the mathematicians viewed intuition as a "necessary component for developing knowing" (p. 31). Burton continued that while these mathematicians had this opinion, none of them offered comments on how one might develop mathematical intuition.

Methods

This study took place at a large doctoral-granting research institution in the United States. Participants were recruited via email. Nine mathematicians (five pure mathematicians and four applied mathematicians) volunteered to participate and were interviewed by the author. In the interviews, participants were asked if they had ever used the term mathematical maturity. If the participant indicated that they had, they were then asked several probing questions about the nature of the term. If they participant indicated that they had not ever used the term mathematical maturity, they were asked if they were familiar with the term. If a mathematician was not familiar with mathematical maturity, the interview was terminated.

For the mathematicians familiar with the term mathematical maturity, the interviewer asked 1) in what context they had used the term mathematical maturity, 2) if mathematical maturity is a feature of a person or a mathematical artifact, 3) how they would identify if a student is mathematically mature, 4) what features of a student (or their work) would help them to identify the student as mathematically mature, and 5) if any of the features are clear or key signs of mathematical maturity. Finally, the interviewer asked the mathematician if mathematical maturity is something they aim to foster in their students and if so, in which classes, why those classes, and how they attempt to foster mathematical maturity in their students.

Mathematicians were not compensated for their participation. Interviews ranged in length from 2 minutes to 50 minutes. Mathematicians varied in years of experience teaching advanced mathematics courses and areas of study.

Analysis

Each of the interviews in which the mathematician indicated familiarity with the term mathematical maturity was transcribed and analyzed. The data was analyzed using open coding in the style of Strauss and Corbin (1990). Each interview was individually coded for descriptions of mathematical maturity and various aspects or indicators of mathematical maturity offered by the participant. Categories of these indicators and descriptions were then tentatively identified per interview. The various themes and indicators of mathematical maturity of each of the interviews were then synthesized to identify categories of codes discussed by multiple mathematicians. Transcripts were then reviewed for any additional occurrences of the codes not identified in the earlier pass through the data.

Preliminary Results

Pure Mathematicians and Applied Mathematicians

One striking finding from this study concerns the interviews with (self-reported) applied mathematicians. As noted in the methods, each interview began with the question, “Have you ever used the term mathematical maturity?” Surprisingly, none of the four applied mathematicians has used the term. Moreover, when probed further, none of the four applied mathematicians even appeared to be familiar with the term. At best, two of the four conjectured that the concept was related to a student’s competence in mathematical activities, but each of the applied mathematicians did not feel comfortable continuing the conversation around mathematical maturity. As such, the remainder of the results focus solely on the interview data from the pure mathematician participants.

Aspects of Mathematical Maturity

Below is a table representing each of the codes, and the categories they were sorted into, that resulted from the open coding analysis described above. As seen in Table 1, each code included was present in at least two different interviews and the codes were sorted into three categories: Ways of thinking about mathematics, Mathematical intuition, and Comfort with and competence in mathematics. Due to constraints on the length of this proposal, I note that the Ways of thinking and Mathematical Intuition categories are closely tied to the literature on mathematical beliefs and mathematical intuition (respectively) and provide only a brief description and selected quotes to highlight some of the codes included in the Comfort with and Competence in Mathematics category.

Table 1. Categories and codes for mathematical maturity mentioned by the pure mathematicians.

	M2	M3	M4	M5	M6

Ways of Thinking about Mathematics					
Having a holistic view of mathematics	**X**		**X**	**X**	
Being accepting multiple representations and changing definitions		**X**	**X**	**X**	**X**
Having autonomy or agency over their own learning	**X**		**X**		
Mathematical Intuition					
Knowing what to do with problems		**X**			**X**
Recognizing the crux of an argument		**X**			**X**
Knowing if a solution makes sense	**X**				**X**
Comfort with and Competence in Mathematics					
Having the ability to absorb and use definitions and theorems		**X**		**X**	
Having the ability to effectively communicate mathematics		**X**	**X**	**X**	**X**
Having the ability to abstract and make connections across topics			**X**	**X**	**X**
Having the ability to self-assess, validate, and reconstruct arguments	**X**				**X**

Comfort with and competence in mathematics. This category largely focuses on necessary skills described as indicators of mathematical maturity by the pure mathematicians in the study. For instance, M5 believed the ability to effectively communicate mathematics was essential to one's mathematical maturity. When asked what a specific indicator of mathematical maturity might be, M5 said, "Being able to take an intuitive idea and express it using a sensible notation, and yeah, putting it into words in a sense". M5 continued to explain that being able to express one's ideas, as well as understanding the ideas of others', regardless of the other person's level of comfort with mathematical notation is essential.

When asked if he had used the term mathematical maturity, M3 immediately agreed and continued to explain that mathematical maturity is the ability to abstract in a useful way. As an example, M3 said "if we show our students a proof that there are infinitely many primes, what's the point? It's not really that there are infinitely many primes." Later he explained that by abstraction, he meant "the ability to read the story and understand the moral rather than just seeing that the tortoise beats the hare". Throughout his interview M3 focused on students' abilities to view (or abstract) the bigger picture and motivation behind a proof as indicative of mathematical maturity.

Conclusion

As a preliminary study, these findings are still being interpreted. The findings of this study will also have various limitations due to the small sample size and exploratory nature.

However, mathematical maturity is a concept that has largely been deemed ineffable yet continues to be used in mathematical practice. Moreover, we see that not all mathematicians (notably the applied mathematicians in the study) are familiar with the term. Meanwhile, as mathematical maturity, mathematical beliefs, and mathematical intuition are intrinsically tied, not only to each other, but also to student success, this study aims to provide an empirical first step toward understanding mathematical maturity. Future research considering these topics could lead to future strides for mathematics education research. Such a clearer conception of mathematical maturity can afford future research on fostering and developing mathematical maturity over time and measuring a learner's mathematical maturity.

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