Conversations on Density of $\mathbb{Q}$ in $\mathbb{R}$

Ofer Marmur
Simon Fraser University

Ion Moutinho
Universidade Federal Fluminense

Rina Zazkis
Simon Fraser University

We explore the notion of density of the set of rational numbers in the set of real numbers, as interpreted by undergraduate mathematics students. Participants' responses to a scripting task, in which characters argue about the existence of one or infinitely many rational numbers in a real number interval, comprise the data for our study. The framework of reducing abstraction is used in explaining the participants' mathematical behavior when coping with the task. The analysis reveals informal ideas related to density as well as unconventional understandings of density-related concepts of rational numbers and infinity.

Keywords: density, rational numbers, scripting tasks, reducing abstraction

The notion of density is one of the main characteristics of the rational numbers, which distinguishes these from natural numbers and integers. However, the notion of density has not yet received significant attention within the growing body of research in mathematics education at the tertiary level. While the notion of density is the main focus of this paper, we demonstrate how engaging students in a discussion on density brings to light some of their underlying ideas on the structure and nature of rational numbers. However, prior to presenting the details of our study, we supply an overview of the notion of density in mathematics education research, followed by a discussion on mathematical nuances related to the concept.

The Notion of Density in Mathematics Education Research

The investigation of learners' understanding of the notion of density in prior research was associated with the development of understanding of rational and irrational numbers. In this regard, Vamvakoussi & Vosniadou (2004) argued that the understanding of rational numbers requires a conceptual change, which is a lengthy and gradual process. They further assumed that the idea of discreetness, developed through experience with natural numbers, is a "fundamental presupposition which constrains students' understanding of the structure of the set of rational numbers" (p. 457).

In studies that focused on learners' ideas in relation to density, middle and high school students were often given a particular interval (such as "numbers between 0.21 and 0.22" or "numbers between 1/10 and 1/11"), and subsequently asked multiple variations of similar-idea questions – such as whether there exist any rational numbers in the interval, how many rational numbers exist in the interval, and so forth (e.g., Vamvakoussi & Vosniadou, 2004, 2007; Vamvakoussi, Vosniadou, & Van Dooren, 2013). The findings pointed to a natural number bias, in the sense that the discreetness of natural numbers, as well as the existence of a successor in natural numbers, were extrapolated to rational numbers. This resulted in frequent mistakes, reported both with common fraction and decimal fraction representations of rational numbers.

In several studies that explored teachers' understanding of irrational numbers, the issues related to density appeared as part of the tasks. For example, Sirotic and Zazkis (2007) focused on the density of both sets of rational and irrational numbers, and inquired into how prospective secondary teachers' "fit together" these two sets. In particular, they asked participants to determine whether it was possible to find a rational (or irrational) number between any two rational (or irrational) numbers. We note that in the density related items there was no specific attention to the option of a general interval of real numbers, that is where one endpoint may be rational and the other irrational. In the current paper we address this aspect and attend to the more general property – the density of the set of rational numbers in the set of real numbers. The following section elaborates on this issue.

On Density: Density of $\mathbb{Q}$ vs. Density of $\mathbb{Q}$ in $\mathbb{R}$

We observed that most of the studies that explicitly discuss the density of rational numbers attend exclusively to the set of rational numbers. However, the notion of density of the rational numbers is more general: not only that the set of rational numbers $\mathbb{Q}$ is dense, i.e., dense within itself, but it is also dense in the set of real numbers $\mathbb{R}$. Formally, we attend to the following definitions:

- Definition 1: Given a $X \subset \mathbb{R}$, we say that X is ***dense*** if for every $a, b \in X$ there is a $c \in X$ such that $a < c < b$.
- Definition 2: Given a subset $X \subset \mathbb{R}$, we say that X is ***dense in*** $\mathbb{R}$ if for every $a, b \in \mathbb{R}$ there is a $c \in X$ such that $a < c < b$.

Note that Definition 2 appears in formal mathematics texts (e.g., Bartle & Sherbert, 2011; Courant & John, 2012), while variations of Definition 1 are implied in the mathematics education research literature (e.g., Vamvakoussi & Vosniadou, 2010; Malara, 2001). That is, mathematics education research has primarily focused on the existence of rational numbers in a *rational* number interval, rather than in the interval of real numbers. However, the density of a set does not imply its density in $\mathbb{R}$. Consider for example the set $X = (0, 1) \cap \mathbb{Q}$, which is dense (meaning within itself), yet not dense in $\mathbb{R}$. Hence, the density of $\mathbb{Q}$ in $\mathbb{R}$ cannot be deduced from the density of $\mathbb{Q}$, and therefore requires a separate consideration. As such, our study attends to the notion of density of $\mathbb{Q}$ in $\mathbb{R}$, specifically as understood by undergraduate students.

Theoretical Framework: Reducing Abstraction

The framework of *reducing abstraction* was introduced by Hazzan (1999) when inquiring into students' struggles with concepts and ideas of Abstract Algebra. The basic premise of the framework is that when solving mathematical problems, students may operate on a lower level of abstraction than is intended by the task or the instructor. The framework is based on three different interpretations of abstraction discussed in the literature, described briefly below. It is important to note that these interpretations are neither mutually exclusive nor exhaustive.

a) The interpretation of abstraction level as the *quality of the relationship between the object of thought and the thinking person* is based on the idea that abstraction is not a property of an object, but rather on "a property of a person's relationship to an object" (Wilensky, 1991, p. 198). An illustration of this idea is provided by Noss and Hoyles (1996) who wrote "To a topologist, a four-dimensional manifold is as concrete as a potato" (p. 46).
b) The interpretation of abstraction level as *reflection of the process–object duality* is based on the process–object duality, suggested by several theories of concept development in mathematics education (e.g., Dubinsky, 1991; Sfard, 1991). Despite the differences in further elaborations, researchers agree that during learning stages of a mathematical concept, its conception as a process precedes – and as such is on a lower level of abstraction – than its conception as an object.
c) The interpretation of abstraction level as the *degree of complexity of the mathematical concept* is based on the assumption that a more complex object is more abstract. For instance, a particular example demonstrating a property is less abstract than a general

claim justifying a property; a particular element of a set, or a particular subset, is less abstract than the set itself; and so forth.

In addition to the initial work in Abstract Algebra (Hazzan, 1999), the framework was employed in different areas of mathematics, such as differential equations (Raychaudhuri, 2014) and a variety of topics in school mathematics (Hazzan & Zazkis, 2005). Hazzan (2003) provided a comprehensive report that illustrated the application of the reducing abstraction framework in a variety of situations and topics taken from undergraduate mathematics. In this paper we describe an application of the framework in analyzing students' ideas of density, and demonstrate the role of reducing abstraction in students' conceptions of real and rational numbers.

The Study

Participants and Setting

The participants of the study were 95 first-year undergraduate students enrolled in a Bachelor's degree in mathematics in a highly-ranked university in Brazil. At the time of data collection the students were enrolled in a "Foundations of Mathematics" course, which provided a foundation for subsequent Pre-Calculus, Calculus, and Real Analysis courses. It was assumed that the students were familiar (at least to some degree) with how rational numbers are defined, with different representations of rational numbers, and with the relation between different number sets (natural-, integer-, rational-, irrational-, and real numbers). During the course, special attention was given to the representation of numbers and intervals on the real number line. In the middle of the course, the students responded to a task that dealt with the notion of density, as described in the following section.

The Task and Research Questions

The task that was presented to the participants of the study belongs to the genre of scripting tasks. In such tasks, participants are typically given a beginning of a dialogue, referred to as a prompt, and are asked to extend the dialogue in a way they find mathematically and pedagogically fit. Scripting tasks were used in prior research in various mathematical contexts (e.g., Kontorovich & Zazkis, 2016; Marmur & Zazkis, 2018; Zazkis & Herbst, 2018), and their advantages were elaborated upon in detail (e.g., Zazkis, 2018). In particular, a significant feature of scripting tasks is that they provide script-writers the opportunity to consider or revisit the mathematical ideas related to the task, and offer researchers a lens on the script-writers' understanding of these particular mathematical concepts and relations.

The prompt for the particular task analyzed herein (see Figure 1) presents a disagreement on how many numbers can be found in a given interval of real numbers.

Pedro:	Hello, Maria! Did you manage to explore the applet[1]?
Maria:	Yes, it was quite nice. Here's my conclusion: Given two distinct numbers on the line, a and b, we can always find a rational number between a and b.
Pedro:	Wow, my conclusion was very similar to yours, but there is a difference. See: Given two distinct numbers on the line, a and b, there are infinitely many rational numbers between a and b.
Maria:	I don't think so, how did you come to that conclusion?
Pedro:	...

Figure 1: Prompt for the scripting task

[1] The applet (https://www.geogebra.org/m/nruYwQAd) provided an interactive and virtual environment to explore density. However, students' interaction with the applet is outside the scope of our analysis in this paper.

In addition to continuing the dialogue (Part-A of the task), the participants were asked to present a mathematical analysis reflecting their personal understanding of the issue (Part-B). This was in order to be able to distinguish between student-character statements that might represent a "student way of thinking", and statements that represent the script-writer's own ideas.

The task was designed to uncover the participants' informal ideas about the density of $\mathbb{Q}$ in $\mathbb{R}$, ideas on which the formal proof is built in a later course. Note that while the claims of Maria and Pedro are presented in a form of disagreement in the task, they are in fact equivalent as each claim implies the other.

Initially, the task was designed to address the following research question:

- What is revealed in the participants' claims in regard to their informal ideas about the density of $\mathbb{Q}$ in $\mathbb{R}$?

Through the examination of data, we added another research question, to which we attend herein:

- What is revealed in the participants' claims in regard to their understanding of infinity, as well as real and rational numbers?

Data Analysis

The data for this study are comprised of the scripted dialogues composed by the participants, together with their personal mathematical analyses of the issues at hand. As in prior research that used script-writing for data collection, we regarded the ideas expressed in the scripted dialogue, on which both characters agree, as ideas held by the student who composed the dialogue, unless explicitly stated otherwise in the mathematical analysis section.

In the first round of analysis we identified with which character (Maria or Pedro) the script-writers agreed. In the second round we focused on the arguments that were provided in support of one of the characters' views. While focusing on the existence of rational numbers in an interval, the participants revealed in the voices of their characters some unconventional understandings of rational numbers and ideas related to infinity, which are in discord with mathematical convention. Accordingly, in the third round of analysis we identified and analyzed these unconventional and at times idiosyncratic understandings by utilizing the framework of reducing abstraction (Hazzan, 1999). The findings from this round are presented below.

Findings

While the instruction of the task did not require the students to choose which statement they thought was correct, most of the participants explicitly agreed with one of the characters in the dialogue. In fact, out of the 95 participants, the majority (n=69) sided with Pedro. The other students either agreed with Maria (n=10), or with both (n=11), or did not voice any explicit agreement with either character (n=5). However, regardless of the chosen claim (Maria's or Pedro's), the students' arguments and justifications were at our focus of attention, as they provided a lens into their understanding of density and related concepts. In what follows we exemplify participant ideas related to density, though at times incomplete or erroneous, that illuminate their understanding of real and rational numbers.

Referring to a Ruler to Spread Rational Numbers on the Number Line

One method students employed in order to deal with the task was to first "spread" rational numbers all over the number line, typically represented with a ruler, and only subsequently place the points a and b in their accurate location, whilst already having rational numbers "ready-made" in between.

Pedro: Don't you know that between two points on the number line we have several other points?

Maria: Yes, I know! But I don't agree that there are infinite numbers.

Pedro: I'll explain with a ruler how I came to this conclusion and you're going to agree with me. When we get the school ruler we can see the cm because we have the traces, right? So we can also do with millimeters.

Maria: Yeah. But what does this have to do with what I said?

Pedro: Calm down, I'm getting there! After we have observed that between the cm exists the mm, and that to arrive at the value of 1 cm we need to count 10mm, then we can conclude that in order to arrive at the value of 1 mm, we will need to count another 10 of some value that we do not use normally and so on. As you can see, my points A and B are between 0 and 1, and when we partition that same measure we realize that there can be found infinite numbers between them. The more partitioned, the more numbers are found!

We regard this type of mathematical behavior as reducing the level of abstraction in the following three ways. First, we recognize this abstraction reduction as reflection of the process-object duality (Hazzan, 1999). That is, the students attend to the process of *creating* infinitely many rational numbers using "smaller and smaller" partitions, rather than to the *existence* of these numbers. We note that the above excerpt does not actually demonstrate the existence of infinitely many rational numbers between a and b, but only points towards a process that can continue indefinitely in order to produce them.

Secondly, we view the abstraction level in regard to the applicability, concreteness, and tangibility of the mathematical object. In this case, the rational numbers are related to a real-life application of measuring distances, and exist in a physical form as lines on the measuring ruler. Thirdly, we consider the abstraction reduction in relation to the logical complexity of the given statement. Meaning that instead of demonstrating the existence of rational numbers (whether one or infinitely many) for a *given* segment, the students herein swap the logical order by first creating rational numbers with a ruler, and only then positioning the segment on the number line. This mathematical behavior is in line with the logical difficulties observed by Dubinsky and Yiparaki (2000), where students confuse between AE and EA statements (i.e., $\forall x\ \exists y\ R(x, y)$ versus $\exists y\ \forall x\ R(x, y)$).

Particular Intervals and Sequences with Discernable Patterns

Many students chose to work with specific intervals in which rational numbers were searched for (that is, with particular choices for a and b), typically accompanied by a construction of a sequence with a clear pattern. The following excerpt illustrates this tendency:

Pedro: Now we can take as an example a number that is between 0 and 1, tell me all that comes to your mind.

Maria: Well, we can think of half of 1 = 1/2 = 0.5.

Pedro: Yes, we can, this number is certainly between 0 and 1 right there in the middle. But we can get a lot more numbers. Think of a few more.

Maria: Okay, how about these: 1/3, 1/4, 1/5.

Pedro: Perfect, those are certainly between zero and one. Did you notice that you can increase the denominator until you get tired?

Similarly, other sequences in various participants' scripts followed an easy-to-guess pattern, such as the sequence 0.11, 0.101, 0.1001, 0.10001, …, given between 0.1 and 0.2. Most sequences approached one of the endpoints of the interval (in fractional or decimal representation), though some were placed somewhere “in between”, e.g., the sequence 0.1, 0.11, 0.111, 0.1111, …, in the interval $(0,1)$. This demonstrates an abstraction reduction towards the process (versus object), where students focus on the calculative aspect of producing particular sequences of rational numbers in an interval. Furthermore, we suggest that by producing simple-patterned sequences as illustrated above, the students were not attending to the arbitrary nature of the segment (a, b), and how rational and irrational numbers are situated in it.

Additionally, the level of abstraction is reduced here in relation to the degree of complexity of the concept of thought (Hazzan, 1999). Not only is there a preference towards particular numbers rather than arbitrary real a and b, but also a and b are always chosen as integers or rational numbers, thus reducing the complexity degree of the concept of an interval. Consequently, the level of abstraction being reduced is also manifested by students accepting particular examples as a valid justification (see Hazzan & Zazkis, 2005). In most cases we could find no evidence, neither in the scripts nor in the accompanying mathematical analysis, which demonstrated awareness that the particular examples were not generic, in the sense that the general case could not be concluded from the chosen examples. To the contrary, we witnessed cases in which the consideration of segments with irrational endpoints was explicitly rejected, demonstrating that working in a reduced level of abstraction was a conscious choice.

Pedro: I imagined A and B as integers…

Maria: But, does it work for my numbers? Is this a rational number that I find between A and B?

Pedro: I'm not sure. I think that for this rational number to be the midpoint it is necessary for A and B to be rational numbers. Imagine if the points were $\sqrt{2}$ and π. I think the midpoint would be irrational because $\sqrt{2}$ and π are irrational.

Maria: Ihh! It's already complicated. Let's stay with rational numbers for now?

Fractions are Small Numbers

As illustrated in the previous sections, we noticed that many students not only chose to work with specific intervals, but also situated the problem around the number zero. This led us to suspect that some students have a concept image (e.g., Tall & Vinner, 1981; Vinner, 1983) of fractions as “small numbers”, that is, what we refer to as positive proper fractions. The following representative excerpt supports this interpretation, exemplifying only positive proper fractions without attention to the interval in which rational numbers are being sought:

Pedro: Note that if I divide a unit into 2,3,4,5 parts and get one of them, ex. $\frac{1}{2}, \frac{1}{3}$, and so on ... I'm dividing this unit into smaller and smaller parts but I'll never get to zero. And as I can put any integer value, there will be infinite parts without reaching zero.

Maria: I had not thought of it this way, but that does not mean that my statement is wrong.

Pedro: Yes, I agree with you that we will always find a point between a and b. But my demonstration goes further and shows that we can find infinite points between a and b.

As Raychaudhuri (2014) elaborated, students can reduce the abstraction level of a problem or concept by ignoring or “freeing” the context in which it is situated. In the current case this is done by attending to rational numbers with no regard to the segment (a, b) in which they are to be found. Our interpretation of this tendency as illustrative of students’ abstraction reduction is further supported by Zazkis (2014), who regarded the students’ evoked example space (Watson & Mason, 2005) – which “accounts for what specific examples are actually used” (Zazkis, 2014, p. 34) – as indicative of how students reduce the abstraction level of a concept by attending to particular examples.

Personal Meaning of "infinite rational numbers"

Another phenomenon we observed in the data was students' preference towards their own personal meaning of mathematical concepts over conventional interpretations. Note that the Portuguese formulation of the task "infinitos números racionais entre a e b" literally translates to "infinite rational numbers between a and b", though for the purpose of this report was translated to "infinitely many rational numbers between a and b". However, some students interpreted the expression "infinite rational numbers" as a rational number that has an infinite decimal representation, rather than the intended meaning of an infinite amount of rational numbers. Once such a number was found (e.g., 0.666…), the subsequent conclusion was that Pedro's assertion was correct, and therefore there are "infinite rational numbers" between a and b.

Related to Hazzan (1999) interpretation (a) above, Raychaudhuri (2014) found that one way in which students reduce the level of abstraction is by referring to their own personal meaning of a concept. That is, they choose their own interpretation, which is based on their personal mathematical (and non-mathematical) experience, rather than search for and base their ideas on conventional mathematical meanings. By regarding 0.666… as "infinite numbers", the script-writer reduced the abstract nature of grasping a non-concrete infinite amount of numbers, and changed the meaning to a single and concrete number whose digits continue indefinitely.

Discussion and Conclusion

The current research was designed to gain deeper insight into undergraduate students' understanding of the density of rational numbers within the set of real numbers. The findings demonstrate the complexity of this notion for learners. In particular, the participants in this study demonstrated difficulties in justifying their chosen mathematical claims in an appropriate manner. This revealed unconventional yet somehow limited understandings of the relation between rational and irrational numbers, as well as the notion of infinity.

When analyzing the data, the framework of reducing abstraction proved to be a valuable tool in explaining the participants' mathematical behavior and their coping mechanisms with the task. Rather than attending to the general structure of a segment on the real number line, and how rational and irrational numbers interlay within it, it seems that the participants concentrated on specific examples, contexts, processes, and personal meanings, consequently reducing the intended abstraction level of the task. This also revealed certain mathematical conceptions and ideas held by the participants: a concept image of fractions as small (positive) numbers; a restricted view on the notion of infinity which is solely regarded as a process (e.g., Dubinsky, Weller, Mcdonald, & Brown, 2005); and a rational-number bias in the sense of: (a) a strong preference towards working with rational numbers whilst rejecting cases with irrational numbers, and (b) regarding particular examples with rational numbers as explanatory justifications of the general case which includes real numbers as well.

In conclusion, the contributions of our findings are twofold. First, our study expands on previous research in mathematics education, and explores not only learners' understanding of the density of $\mathbb{Q}$ (within itself), but also of the density of $\mathbb{Q}$ in $\mathbb{R}$. The findings suggest that by placing the discussion in the context of real numbers rather than rational numbers only, the level of mathematical complexity rises, which may also explain the resulting student behavior of reducing the level of abstraction. Secondly, when examining the scripts that are situated in this more general mathematical context, the findings demonstrate mathematical ideas that are held by learners not only in relation to rational numbers, but also in relation to irrational numbers and the notion of infinity. These insights into unconventional student understandings could in turn be utilized for the development of suitable teaching practices that address these student conceptions.

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