Monster-barring as a Catalyst for Connecting Secondary Algebra to Abstract Algebra

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This proposal reports on a teaching experiment in which a pair of prospective secondary mathematics teachers leverage their knowledge of secondary algebra in order to develop effective understandings of the concepts of zero-divisors and the zero-product property (ZPP) in abstract algebra. A critical step in the learning trajectory involved the outright rejection of the legitimacy of zero-divisors as counterexamples to the ZPP, an activity known as monster-barring *(Lakatos, 1976; Larsen & Zandieh, 2008). This monster-barring activity was then productively repurposed as a meaningful way for the students to distinguish between types of abstract algebraic structures (namely, rings that are integral domains vs. rings that are not). The examples of student activity in this teaching experiment emphasize the importance of identifying, attempting to understand, and leveraging student thinking, even when it initially appears to be counterproductive.*

Keywords: Teaching experiment, Zero-product property, Monster-barring

Introduction

Abstract algebra is seen as an important course in the mathematical preparation of secondary teachers, largely because of its potential to enable students to view the familiar content of secondary algebra through a more advanced lens. For example, it is recommended that prospective teachers come to regard the secondary algebra that they will be teaching as "the algebra of rings and fields" (CBMS, 2012, p. 59). Thus, in light of a significant body of literature reporting that students struggle to view secondary content from such an advanced persepective (e.g., Wasserman, 2016; Wasserman et al., in press; Zazkis & Leikin, 2010), a productive avenue of insight is to investigate student thinking about the algebraic properties that characterize such fundamental structures as rings, integral domains, and fields. To this end, the research question that motivated this study was: how might prospective secondary teachers preparing to take abstract algebra be able to adapt their existing understandings of an algebraic property to be effective in abstract algebra?

To answer this question, I conducted a teaching experiment (Steffe & Thompson, 2000) with a pair of prospective teachers preparing to take an introductory course in abstract algebra. The purpose of the teaching experiment was to investigate how prospective teachers might "assimilate their understanding of secondary mathematics with advanced mathematics" (Wasserman, 2017, p. 199) by focusing on: (i) student thinking related to the zero-product property (ZPP), a tool for solving equations in secondary algebra and the definitive characteristic of integral domains in abstract algebra, and (ii) how such thinking might be leveraged to enable students to develop an effective understanding of the ZPP in abstract algebra.

Literature and Theoretical Framing

With respect to my research question, I employed Thompson's (2008) tools for *conceptual analysis* in order to describe the characteristics of productive understandings of the ZPP in abstract algebra. To this end, a *way of understanding* is a meaning or conception that a student has for a particular mathematical idea (Harel, 1998). A way of understanding might include a system of strategies, analogies, informal descriptions, and examples and non-examples. Harel (1998) proposed that a student holds an *effective* way of understanding a mathematical idea if, in addition to retaining that way of understanding over time, she is able to:

- Criterion I: reformulate and articulate it in her own words,

- Criterion II: think about it in a general way, and
- Criterion III: coordinate it with her ways of understanding other ideas.

These criteria provide an observable way to determine if a student holds an effective way of understanding, but it remains unclear exactly what these criteria mean for zero-divisors and the ZPP in an abstract algebra setting. While criterion I – the student's ability to formulate the concept in her own words – is relatively straightforward, in order to operationalize Harel's criteria it is necessary to specify what it means for a student to think about zero-divisors and the ZPP in a general way (criterion II), and also to incorporate her thinking about other concepts (criterion III).

In order to operationalize[1] criterion II – what it means to think about a concept in a general way – I adopted Alcock and Simpson's (2011) perspective that classification of examples is a fundamental mathematical task. Indeed, a fundamental task for introductory abstract algebra students is to determine if a new example structure is an integral domain, which essentially amounts to determining whether the structure contains zero-divisors. Though the ability to consistently classify examples is rarely the final objective, it can be a useful opportunity for students to gain some initial experience with the underlying concept (e.g. Ross & Makin, 1999). Particularly, students with a way of understanding that is not fully developed will probably be unable to use it to consistently classify examples (e.g. Davis & Vinner, 1986). Thus, I used the ability to consistently classify algebraic structures on the basis of a particular property as evidence that a student was thinking about that property in a general way.

Methods

I adopted the teaching experiment methodology (Steffe & Thompson, 2000) as a means of exploring and refining the conceptual analysis – that is, the characterization of an effective way of understanding the ZPP and my hypothesis about how students might come to achieve such a way of understanding. I conducted the teaching experiment reported here with two undergraduate students, Brian and Julie (both pseudonyms), who were both beginning the first semester of their junior years at a small, public liberal arts college as mathematics education majors and prospective secondary mathematics teachers. Both had completed a course in linear algebra (both earning B's) but had not yet taken an introduction to proof course. This was typical for mathematics education majors at this particular institution, who instead were required to take an 'abstract algebra for future secondary teachers' course that focused more on the relevance of abstract algebra to secondary algebra than on the rigors of proof. Both Brian and Julie were preparing to begin this course when they participated in this study.

The teaching experiment consisted of 4 sessions lasting between 75 and 90 minutes each; I served as the teacher-researcher for all sessions. Each session was recorded with LiveScribe pen technology, which records the students' pen strokes with synchronized audio (called a pencast). I constructed models of Brian and Julie's ways of understanding using *on-going* and *retrospective* analysis techniques (Steffe & Thompson, 2000). The instructional tasks of the teaching experiment centered on solving equations, a mathematical activity that is familiar to students from school algebra that can serve as a useful means of gaining insight into the algebraic structures – like groups (e.g. Wasserman, 2014) and rings (e.g. Cook, 2014) – that form the foundation of abstract algebra.

[1] Here I will only explicate criterion II, as the excerpts of student activity relevant to criterion III were trimmed to comply with space constraints.

Results

Though it is beyond the scope of this brief proposal to comprehensively document the students' entire learning trajectories, here I will present and analyze the key episode of the teaching experiment in which Brian's outright rejection (i.e. *monster-barring* – see Lakatos, 1976; Larsen & Zandieh, 2008) of zero-divisors was repurposed in order to classify algebraic structures in a way consistent with how experts distinguish between integral domains and rings that are not integral domains.

Monster-Barring Zero-divisors in $\mathbb{Z}_{12}$

At this point in the teaching experiment, Brian and Julie had correctly solved several equations in $\mathbb{R}$, including $4x = 0, 4(x-5) = 0$, and $(x+2)(x+3) = 0$. I encouraged them to solve the same equations in $\mathbb{Z}_{12}$, hoping that they would notice the presence of multiple solutions and ultimately identify the failure of the ZPP as the cause. But, just as in $\mathbb{R}$, they both asserted that $x = 0$ is the only solution to $4x = 0$ and $x = 5$ is the only solution to $4(x-5) = 0$ in $\mathbb{Z}_{12}$, with Brian specifically mentioning that "the only way for 4 times a number to equal 0 is by multiplying by 0." Similarly, Julie's solution to solving $(x+2)(x+3) = 0$ in $\mathbb{Z}_{12}$ employed what appeared to be the ZPP and proceeded almost identically to her response to the same equation in $\mathbb{R}$, the only difference being that her solutions were $x = 9$ and $x = 10$ (instead of $x = -2$ and $x = -3$). Brian's response made it clear that he also did not detect any differences between $\mathbb{R}$ and $\mathbb{Z}_{12}$:

Brian:	Uh … what was the point of that?
Researcher:	What was the point of what?
Brian:	That is literally the exact same as normal math.
Researcher:	OK, so … [laughs]. OK, so I want to break this down. What is, what is that? What are you … what is the same as normal math?
Brian:	The way she solved it with $\mathbb{Z}_{12}$ is the exact same way you solve that in normal factoring.

Simpson and Stehlikova (2006) proposed that, in cases in which students struggle to identify critical aspects of an algebraic structure for themselves, instructors should "explicitly guide attention to, first, those aspects of the structure which will be the basis of later abstraction" (p. 368). As my efforts to guide their attention to zero-divisors implicitly via task design were unsuccessful, I decided to heed these recommendations and explicitly point out an instance of zero-divisors. Specifically, referring to the task in which Brian and Julie had proposed that $x = 5$ was the only solution to $4(x-5) = 0$ in $\mathbb{Z}_{12}$, I asked them about the possibility that $x - 5 = 3$ (see the excerpt below) so that they might recognize that $4(x-5) = 4 \cdot 3 = 0$. I phrased my inquiry somewhat unconventionally in terms of the element $x - 5$ (as opposed to simply offering $x = 8$ as an additional solution) because I wanted to maintain focus on the equation's product structure and, potentially, the ZPP. Julie immediately realized (and accepted) that they had overlooked such cases, remarking that she had stopped looking for solutions after identifying $x = 5$ because she had only expected one solution. Brian, on the other hand, rejected the possibility of additional solutions:

Researcher:	What do you think, Brian, you don't look, you've got a skeptical look on your face.
Brian:	I still think that this [motions to $4 \cdot 0 = 0$] is 0, right, but this …
Researcher:	So, can you say what you're pointing to right now?
Brian:	The 4, um, as long as $x = 5$, then that's 0, and I think that's the only way

to 0. This is some type of convoluted plan or a scheme you've come up with. There's no way that this is a 0.

Brian's outright rejection of zero-divisors surprised me – I had predicted that he would react like Julie and reluctantly concede that he had overlooked several solutions (which would then have been an opportunity to encourage them to revisit their rule and whether or not it holds in $\mathbb{Z}_{12}$). Instead, however, I decided to explore Brian's reasons for rejecting (the additional solutions created by) zero-divisors. My first conjecture was that perhaps the clock arithmetic metaphor from the initial task that introduced $\mathbb{Z}_{12}$ was influencing Brian's thinking. Perhaps, for example, he viewed $4 \cdot 3$ as 12, and, as a result, did not identify 12 with 0.

Monster-Barring Zero-divisors in $M_2(\mathbb{R})$

To test this conjecture, I shifted to another example structure, thinking that, if Brian raised no objection to zero-divisors in the new context, then I could conclude that the nature of his previous objection was context-specific to $\mathbb{Z}_{12}$. If, however, he maintained his objections, this would indicate that he was potentially objecting to idea of zero-divisors altogether. I chose $M_2(\mathbb{R})$ as the new example structure because it contains zero-divisors, and it would have been familiar to Brian from linear algebra, thus leaving him with fewer reasons to doubt its legitimacy[2]. I asked if their rule held in $M_2(\mathbb{R})$, and Julie, who seemed relatively unperturbed by the presence and effects of zero-divisors in $\mathbb{Z}_{12}$, drew an analogy with $\mathbb{Z}_{12}$ and seemed to accept the possibility of such elements in $M_2(\mathbb{R})$ (though she was unable to identify any at first), remarking that "when I look back ... there are some other ways to get 0 without multiplying by 0, so I think that maybe there could be a way to multiply two matrices so that you can get the zero matrix." Brian, on the other hand, remained steadfast in his apparent belief that the ZPP was universally inviolable, and responded before even trying to produce a counterexample that "in order to get a zero matrix, you have to multiply by 0." I responded by presenting them with a pair of zero-divisors – specifically, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Brian, after multiplying the two matrices together to obtain the zero matrix, again stood by his original assertion:

Brian:	I don't understand how this example … can count. [sighs]
Researcher:	So why, why wouldn't it count?
Brian:	Because you're still … you still have zeros here. Like you literally just added a 1 somewhere, and said, here you go! It works!
Researcher:	OK, um, when you said 'zeros here,' can … unfortunately, the Livescribe pen can't, uh, can't tell us which ones you're pointing to.
Brian:	OK … these ones [motions to and marks the zeros in $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$]. So there are zeros involved.
Researcher:	There are zeros involved.
Brian:	Yes, so I don't think this should, this should count as an example that we can use. I, I just don't believe that, that this is OK.

Because the nature of Brian's objection in this case was that "there are still zeros involved," I responded by presenting him with a zero-divisor pair that did not involve any 0 entries: $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$. This time, after verifying for himself that the product of these two

[2] It is not completely inconceivable that Brian viewed $\mathbb{Z}_{12}$ as a contrivance that I created purely for the purposes of this teaching experiment. There would be no such concerns with $M_2(\mathbb{R})$.

matrices was indeed the zero matrix, he maintained his skepticism, this time on the grounds that 0 was *not* involved:

Brian:	Um, I'm still skeptical because I still think you need zeros to get zeros, and … you're not multiplying A times, er … you're not multiplying A and B together to get 0, um, because A and B have to be 0.

Brian's refusal to accept zero-divisors in both $\mathbb{Z}_{12}$ and $M_2(\mathbb{R})$ suggests that his reasons for doing so were not context-specific and that he was indeed objecting to counterexamples to the ZPP in a more general way.

Brian's rejection of zero-divisors across algebraic contexts is an example of *monster-barring*. In his seminal text *Proofs and Refutations*, Lakatos (1976) defined monster-barring as the outright rejection of a counterexample on the grounds that it is "a pathological case" (p. 14). Similarly, Larsen and Zandieh (2008), who repurposed Lakatos's methods for mathematical discovery as design heuristics for RME, characterized monster-barring as "any response in which the counterexample is rejected on the grounds that it is not a true instance of the relevant concept" (p. 208). This includes cases in which students summarily reject a counterexample without an apparent reason. Indeed, several of Brian's comments support the assertion that he viewed zero-divisors as pathological and, as a result, he refused to consider them as counterexamples to the ZPP.

Although monster-barring might, at first, seem to be counterproductive and in need of correction via direct instruction, Lakatos (1976) suggested there was potential for such activity to be productively repurposed, commenting that mathematical ideas "are frequently proposed and argued about when counterexamples emerge" (p. 16). Accordingly, Larsen and Zandieh (2008) proposed that having students consider and render judgments about the validity of proposed counterexamples and underlying definitions is a form of informal mathematical thinking that can be leveraged to support the development of more formal mathematical concepts.

Leveraging Monster-barring activity to sort algebraic structures

During this new line of inquiry, I asked Brian to identify exactly which products he objected to in the multiplication table for $\mathbb{Z}_{12}$. He and Julie responded by turning to their multiplication table and circling entries.

Brian:	So 6 times 2, 6 times 4 …
Julie:	6 times 6, 6 times 8, 6 times 10.
Researcher:	So you're just going down …
Brian:	We're just finding the places that … it doesn't look like a 0 needs to be there. Like it's awkward, like it shouldn't be on the multiplication table. So, numbers that multiply … don't look like they multiply together would equal 0, we'll find they do.
Julie:	6 times 4, there's a 0.
Researcher:	OK.
Julie:	So, like, the same thing with, like, 8 times 3.
Researcher:	And that's, so, Brian, that's what you're calling an awkward …
Brian:	Yes.
Researcher:	Like zero showing up in an awkward place?
Brian:	Yes.
Researcher:	Where, where does … what are the non-awkward appearances of 0?
Brian:	The places where 0, the top row and the first column in the table show that every one of those numbers is multiplied by 0 to get 0. Those are the normal ways … to get zero.

Researcher:	Are there, so are there any normal ways that are not in the first row or the first column?
Brian:	No.

This was an important exchange for several reasons. First, Brian used the phrase "awkward ways to make zero" to refer to combinations of elements in which "it doesn't look like a zero needs to be there ... numbers that ... don't look like they multiply together would equal 0." Similarly, "normal ways to get zero" are those involving multiplication by 0. This mirrors the distinction between the ZPP (which is equivalent to the absence of zero-divisors in a ring) and its converse (which always holds in a ring). Second, Julie, who was relatively unperturbed by zero-divisors, was able to quickly operationalize Brian's distinction, as evidenced by her immediate engagement in the task. I interpreted this as a sign that Brian's criteria could be a meaningful way for Brian (and even Julie) to engage with zero-divisors and use them to make distinctions between algebraic structures. This hypothesis shaped my instructional decisions and analysis in the remaining sessions of the teaching experiment, which involved Brian using his 'awkward' distinction as a means of distinguishing between structures with and without zero-divisors.

To further elicit Brian and Julie's thinking about awkward and normal ways to make zero, I designed classification tasks that prompted them to decide if a given structure behaved more like $\mathbb{R}$ or more like $\mathbb{Z}_{12}$ (as they had already concluded that $\mathbb{R}$ contains no awkward ways to make zero, unlike $\mathbb{Z}_{12}$). The first structures they considered were $\mathbb{Z}_{12}$ and $M_2(\mathbb{R})$, both of which they had worked with earlier in the teaching experiment. Brian immediately responded that $M_2(\mathbb{R})$ should be classified as "more like $\mathbb{Z}_{12}$."

Brian:	Definitely $\mathbb{Z}_{12}$.
Researcher:	Why? What makes you so sure?
Brian:	Well, earlier we discussed that $\mathbb{Z}_{12}$ has some awkward ways to make zero and we also talked earlier that the matrices have awkward ways to make zero. Real numbers don't have awkward ways to make zero. So they share that comparison.
Julie:	That does make a little bit more sense because I guess in $\mathbb{Z}_{12}$ three times four is zero. So that would be an awkward way to make zero. You would have to multiply by zero in [the] real [numbers].

Brian's classification of $M_2(\mathbb{R})$ as "more like $\mathbb{Z}_{12}$" suggested that this adaptation to his way of understanding the ZPP might also be generalizable to other contexts. Brian's statements that "$\mathbb{Z}_{12}$ has some awkward ways to make zero" and "the real numbers don't have awkward ways to make zero" are comparable to the more conventional "$\mathbb{Z}_{12}$ contains zero-divisors" and "$\mathbb{R}$ does not contain zero-divisors." Notably, it is not difficult to find superficial similarities between $M_2(\mathbb{R})$ and $\mathbb{R}$: both are uncountably infinite and, moreover, $M_2(\mathbb{R})$ can be viewed as having been constructed from $\mathbb{R}$. The use of Brian's characterization of zero-divisors seemed to supersede such considerations.

Up to this point, Brian had only applied this way of understanding to $\mathbb{Z}_{12}$ and $M_2(\mathbb{R})$, the contexts from which it had emerged in his reasoning, both of which contain zero-divisors. Subsequently, I asked Brian and Julie to classify $(\mathbb{Z}_5, +_5, \cdot_5)$, a structure that, based upon purely superficial characteristics, might be classified as more similar to $\mathbb{Z}_{12}$. However, $\mathbb{Z}_5$ contains no zero-divisors and is thus more similar in this regard to $\mathbb{R}$. Initially, both Brian and Julie hypothesized that $\mathbb{Z}_5$ was more similar to $\mathbb{Z}_{12}$ and $M_2(\mathbb{R})$ because, Brian predicted, "they're [probably] awkward ways to make 0 for $\mathbb{Z}_5$ as well." As they attempted to justify this conjecture by constructing the operation tables, however, they changed their minds:

Julie:	That is more like the real numbers, actually. The only way we ended up getting zero is multiplying by zero. And so that would be more like the real numbers, because in $\mathbb{Z}_{12}$ we could do awkward ways like three times four and get zero. But in the real numbers we have to multiply by zero, and $\mathbb{Z}_5$ also, to get zero.
Researcher:	Do you agree, Brian?
Brian:	I would say it's like the real numbers, yes, after drawing the table out.
Researcher:	And what about the table changed your mind?
Brian:	Looking over, there are no other zeros where other numbers should be, except for where zero is multiplied by another number.
Researcher:	Yeah. I was gonna ask you about that. So I see zeros in the first row and the first column here. Are those not awkward?
Brian:	No. Those are normal ways to get zero. Multiply by zero.

In the above exchange, both students indicated awareness that the 'normal' ways to get zero are the only such ways – for example, Julie mentioned that "we *have* to multiply by zero … to get zero" and Brian noticed that "there are no other zeros where other numbers should be." This is notable because it demonstrates that both Brian and Julie were able to operationalize the awkward/normal distinction to identify a structure without zero-divisors.

Conclusion

This project addresses the issue that prospective teachers do not see the relevance of their abstract algebra coursework to the secondary mathematics they will be teaching. In response, guided by the tools of *conceptual analysis* (Thompson, 2008), I conducted a teaching experiment (Steffe and Thompson, 2000) that investigated how students might be able to adapt their ways of understanding familiar properties from secondary algebra to be effective in abstract algebra. Focusing specifically on the zero-product property (ZPP), my primary research question was: How might beginning abstract algebra students be able to adapt their existing understandings of the ZPP to be effective in abstract algebra? Though I have not presented the learning trajectory in full here, I did describe and analyze its key component: the repurposing of Brian's monster-barring of zero-divisors.

I believe this study has some implications for thinking about pedagogy in mathematics teaching more broadly. Namely, it provides an example for how students' experiences, even if they seem counterproductive and irrelevant at first, can be leveraged effectively to advance their mathematical thinking in productive ways. I see this as a more specific case of a broader phenomenon – an approach to teaching that builds on students' thinking. Much of the mathematics education literature advocates for such an approach. In fact, these findings were brought to light by applying Steffe and Thompson's (2000) methodological principle that researchers – and, indeed, teachers – should assume that students' behavior is rational and that there is great value in attempting to understand and build upon it. This study adhered to this principle by using Brian's thinking as he engaged with the notion of a zero-divisor. However, even more so, this study indicates that such an approach is possible *even when a students' thinking initially appears to be counterproductive*. This suggests two things to me about instruction in abstract algebra for an audience of secondary teachers. First, abstract algebra instruction can model good pedagogical practices. As was done in this study, using student thinking to develop abstract algebra ideas models good pedagogy. For secondary teachers, learning mathematics in ways that mirror good teaching contributes to their development as teachers. Second, not only can we model good pedagogical practices as abstract algebra instructors, we can also be explicit about this modeling. That is, as instructors, we can draw attention to the ways that we are building on students' thinking in our own classrooms. And, as evident from this study, building on student thinking is possible even in extreme cases, when their ideas appears to be unproductive.

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