Mathematicians' Perceptions of their Teaching

Christian Woods
Rutgers University

Keith Weber
Rutgers University

Recent research in mathematics education has uncovered a host of teaching behaviors that are commonly enacted by instructors of advanced mathematics courses. While these descriptive accounts of math teaching are useful, little investigation has been conducted into the reasons for why these practices are so prevalent. In this study, we interviewed seven mathematicians about regularities that have been observed in the literature on the teaching of advanced mathematics. In this report we discuss whether mathematicians view these findings as accurate (they often did), whether they thought these regularities were productive or problematic teaching practices, and why mathematicians engaged in these teaching practices. We discuss how these themes may elucidate the practices of instructors, and later propose implications of the methods of the present study for changing how advanced math courses are taught.

Keywords: proof-based courses, teaching practices, formal content, informal content

In the last decade, numerous researchers in undergraduate mathematics education have sought to understand the pedagogical practice of mathematicians by observing how advanced mathematics courses are taught (e.g., Artemeva & Fox, 2011; Gabel & Dreyfus, 2016; Fukawa-Connelly et al., 2017; Mills, 2014; Pinto & Karsenty, 2018). The results of these studies give researchers insights into what practices are typical in the teaching of proof-based courses. Despite our advances in knowledge about what kind of moves and habits are used by instructors of these classes, little investigation has been carried out into mathematicians' motives and reasons for adopting the practices that are identified specifically with respect to these findings. (Other scholars have investigated mathematicians' motives and rationality for teaching advanced mathematics in general—e.g., Alcock, 2009; Hemmi, 2010; Nardi, 2008; Weber, 2012).

The aim of this study is to shed light on this largely unexplored area of pedagogy by entering into a conversation with mathematicians explicitly about these findings. If we believe that mathematicians are reflective about their own teaching and the teaching that goes on around them, then they have important knowledge to help explain the classroom behaviors that education researchers have documented. We shared the results of some research on teaching in proof-based math courses with mathematicians who have been instructors of these kinds of courses and asked them to reflect on whether these results were an accurate depiction of their experiences, how they felt about the practices described in the results, and what reasons they saw for engaging in or avoiding these practices.

We believe that the contribution of this research is threefold. First, it is a continuation of the dialogue between mathematicians and math educators whose significance several members of our community have extolled (e.g., Iannone & Nardi, 2005; Alcock, 2009). More importantly, it develops this conversation in a direction that has been neglected by recognizing the value that mathematicians bring to interpreting research in which mathematics instructors are themselves the subject of study. Finally, from a practical standpoint, if the education community wishes for its research to effect change in the way that proof-based math courses are taught then how mathematicians feel about the research will suggest different ways of working with instructors to bring about that change. For instance, if mathematicians were surprised and unsettled by the research findings of our field, this suggests a pivotal way toward changing instruction is to disseminate our results to make mathematicians more aware of their teaching practices. However, if mathematicians are aware these teaching practices are common and feel that they are productive or necessary, then understanding mathematicians' rationality for engaging in the teaching practices is pivotal if mathematics educators hope to change them.

Literature Review

Speer et al. (2010) noted the lack of what they referred to as "descriptive empirical research on teaching practice" (p. 100) in collegiate mathematics and called for more work that elaborated the decisions and actions that instructors make when they teach college-level math classes. This sparked an increase in the amount of research that focused on the facets of instruction that are witnessed in advanced math classes (e.g., Fukawa-Connelly, 2012; Pinto, 2013; Gabel & Dreyfus, 2016; Mills 2014). While many researchers focused on case studies (Fukawa-Connelly, 2012; Fukawa-Connelly & Newton, 2014; Lew et al., 2016; Pinto, 2013), other studies analyzed and compared a relatively larger amount of instructors simultaneously. One general finding from this work is that mathematicians' instruction is nuanced and deviates from the "definition-theorem-proof" formalist caricature that is found in the literature (cf. Weber, 2004), but there are nonetheless some commonalities in how mathematicians teach advanced mathematics. Our present research involved presenting the findings of five of these studies (Artemeva & Fox, 2011; Fukawa-Connelly et al., 2017; Paoletti et al., 2018; Moore, 2016; Miller et al., 2018) to mathematicians and asking them to speak about them. We briefly describe these studies here, with a focus on the work by Fukawa-Connelly and his colleagues in 2017 as this is the study that the mathematicians considered in the results we have chosen to include in this report.

Artemeva and Fox (2011) observed 33 college-level math lectures across seven countries with the goal of noticing which elements of instruction were shared by their participants and which elements differed. Prominent among their findings was a pedagogical genre they called "chalk talk," in which an instructor (a) wrote mathematics on the board, (b) narrated aloud what was being written along with her thought processes, and (c) occasionally took a break to present a metanarrative that discussed broader themes with the class. Paoletti and his colleagues (2018) used data obtained from 11 upper-level math instructors' teaching to draw conclusions about the types of questions instructors asked to their classes and how they used these questions to invite participation from the students. Their results showed that instructors often used a large amount of questions per lecture, most of which asked students to provide the next line in a proof, recall a fact, or perform a calculation, but that very often less than three seconds were provided for students to respond to these questions. Moore carried out a task-based study in 2016 to see what considerations went into how four math instructors graded student-written proofs. He found that there was a sizeable variation in the scores that his participants gave to the same proofs and that all of his participants assigned scores to a proof based on what they believed the student was thinking when he or she wrote it. Similarly, Miller, Infante, and Weber (2018) asked nine mathematicians to assign grades to proofs, half of which were designed to contain logical gaps. In addition to confirming Moore's findings, they noticed that several participants assigned less than perfect scores to proofs that they still deemed "correct."

A study performed by Fukawa-Connelly et al. in 2017 sought to clarify the extent to which informal content plays a role in advanced math classes and how instructors present it to their students. Their definition of informal content included any information that could not be conveyed in formal symbolic language, such as heuristics for thinking about a mathematical concept or for producing a proof. In their analysis of the lectures they observed, the researchers identified when informal content was displayed to the class, whether or not the instructor wrote it on the board, and when the content made it into each of the students' notes. They discovered that while informal content is used frequently in advanced math classes, this information is usually only delivered orally and is not written on the blackboard. Moreover, they found that informal content that was only presented orally and not written on the board only appeared in students' notes in less than 3.2% of possible instances. This was contrasted with both formal and informal content that was written on the blackboard, which was almost always found to be recorded in the students' notebooks.

Despite the progress made in detailing widespread regularities in collegiate teaching practice, little has been done to share these results with mathematics instructors and to understand how they make sense of them. As Fukawa-Connelly, Johnson, and Keller (2016) lamented, "there has been little research attempting to explore [the extent of the adoption of reform practices] from the perspective of the instructors who are the ones being asked to change practice" (p. 276). Consequently, this has impeded mathematics education reform efforts as mathematics educators seek solutions to teaching practices that mathematicians do not find problematic. We share their belief that mathematics instructors possess a unique corpus of knowledge that can bring more light to the findings on collegiate teaching than the findings alone are able to convey themselves. The work presented in this report is our attempt to begin a conversation with mathematicians that utilizes this special knowledge and positions the results of mathematics education research according to their viewpoints.

Theoretical Perspective

In the current study, we largely wanted to understand the issue from the perspective of mathematicians. Consequently, we sought to provide accounts of mathematicians' rationality that was grounded in the data that we collected and avoided applying a theoretical perspective on the data at an early stage (Glaser, 1998). Nonetheless, our study was inspired by Herbst and Chazan's (2003) notion of practical rationality and their dictum that teachers do not engage in traditional teaching practices "from a lack of knowledge or a paucity of vision" (p. 3). Rather teachers are reflective and rational; their pedagogical actions are reasoned attempts to fulfill their goals, obligations, and desires, which can involve a complex constellation of disciplinary, institutional, and ethical considerations (Chazan, Herbst, & Clark, 2016). In analyzing our data, we sought to understand what goals mathematicians had and how they thought their goals could best be achieved.

Methods

Participants

The participants for this study were seven mathematicians (one female and six male) from a large, public research university in the northeastern United States. Each participant had taught at least one proof-based mathematics course within the last five years.

Data Collection

Each participant took part in an approximately hour-long semi-structured interview with the first author. These interviews were audio recorded and subsequently transcribed. Questions for the interviews were pre-written in a protocol that focused on each of the five sets of findings described in the literature review of this report. The questions were designed to investigate each mathematician's general impressions of the results, if they believed the results were typical of teaching in advanced math courses, reasons for why they or others engaged in the teaching practices discussed in those results, and strengths and weaknesses of the practices. Follow-up questions were posed by the interviewer to clarify participants' responses or to encourage the participants to expand upon an idea they had shared.

Data Analysis

Data analysis consisted of a separate round of coding for each of the five sets of findings that were the focus of the interviews. Coding of these sections of the interview transcripts was carried out using thematic analysis (Braun & Clarke, 2006). The authors made an initial pass through the data, highlighting excerpts of the participants' responses that exemplified interesting ideas they had shared about the findings that were guiding the discussions. A descriptor for each idea was entered into a Word file along with a more detailed explanation of this idea. After these ideas had all been generated, the authors sought for commonalities among the ideas and arranged them into larger themes that preserved the general spirit of the ideas while capturing the notable similarities between them, along with specific criteria for when an utterance would be coded as a member of that category. For each theme, we sought to understand if mathematicians were expressing that they perceived that they had a goal or obligation to meet in their instruction (cf., Herbst & Chazan, 2003) and if they had a belief about whether a specific teaching practice would be productive or counterproductive for achieving that goal. When these larger themes were created, a second pass through the data was made to code the corresponding sections of the transcripts with them. After, a Word document was created for each of the larger themes and interview excerpts that were coded with that theme were copied and pasted into the corresponding document.

Results

For the sake of brevity, we report only the results pertaining to the portion of our interviews with mathematicians that concerned the portrayal of formal and informal content in upper-level math classes (Fukawa-Connelly et al., 2017). Of the seven mathematicians interviewed, all agreed that the finding that formal content is written down on the blackboard and informal content is usually only spoken orally is an accurate portrayal of advanced math classes. All seven also agreed that these are generally good teaching practices, although five expressed reservations according to a sentiment that good teaching would display a better balance of formal and informal content being written on the blackboard. Five of the mathematicians stated that these findings were generally representative of their own teaching while two denied so, stating that they also often made informal ideas and processes explicit in writing.

During coding, eight broader themes emerged from the interviewees' commentary on these findings. Five of these themes dealt largely with the practice of writing formal content on the blackboard whereas the other three spoke more to the practice of presenting informal content exclusively verbally. In what follows of this results section we describe the ideas that were expressed in these themes, giving examples from interview transcripts to illustrate when appropriate. Each theme is also shown in Table 1, next to the list of mathematicians who had at least one utterance coded with that theme.

Table 1. A table that lists each of the eight themes that emerged from the data, and which mathematicians' utterances comprised those themes.

Mathematicians' rationales for their use or disuse of the blackboard	
Theme	Interviewees that Contributed to this Theme
Blackboard allows for deeper processing and comprehension	M1, M3, M5, M6, M7
Written content is given permanence and importance	M2, M3, M4, M5
Blackboard enables and requires precision	M3, M5, M6, M7
Writing on the board slows the instructor down	M5, M6, M7
Writing formal content emphasizes the language, notation, and nature of mathematics	M4, M7
Oral presentation is needed to hold students' attention	M1, M2, M3, M5, M6
Informal content is conversational in nature	M1, M2, M6
Content should be repeated to be noticed	M2, M3

Writing Formal Content on the Blackboard

Blackboard Allows for Deeper Processing and Comprehension Five of the mathematicians interviewed expressed that writing content on the blackboard allows for it to be carefully processed and aids students in understanding it. As M6 said, "your visual cortex is extremely powerful and somehow seeing words written down on - somewhere, anywhere, a blackboard being a good place, really clarifies a lot of things. So I think writing things down on a blackboard is extremely important."

Within this theme, the mathematicians pointed out that formal content is likely to be less relatable or more unfamiliar to students. M2 said "the technical nitty-gritty is what's least gonna be in the students' minds and so it's most important to have…that written down." On the other hand, informal content is unlikely to need such a high level of processing, and so it is more acceptable to forgo writing it on the board. M5 illustrated this notion when he said "when I'm conveying intuition, the oral words convey that intuition, I don't need to analyze that intuition, intuition is sort of part of the analysis in a way." These mathematicians had the goal that students understand (or at least "process") the technical mathematics; to do so required students having a specific object for this reflection.

Written Content is Given Permanence and Importance Four of the participants spoke to this theme, which deals with two related properties of information that is conveyed on the board. One of these properties is permanence. Writing things such as formal content down on the board preserves them so that they may be checked and referred to later, and so that students can make their own records of them. To this latter end M4 said "what's written on the board students would take notes of, and what's not they might remember, they might not remember. So if you really want to present something you want to write it on the board." The other property that participants mentioned about written content is a higher level of perceived importance of content that is written down compared to content that is not. M5 said "when you write something on the blackboard you are emphasizing that it's important," and M2 remarked that students think "what I have to pay attention to is what's been written down." Implicit in this commentary is that the informal mathematics that is not written down might be less important.

Blackboard Enables and Requires Precision Four of the participants mentioned the significance that the level of precision has on which content is written down during class. Within this theme, the mathematicians noted that formal content is often very precise, and the blackboard is a crucial tool for displaying this precision properly. M3 shared that when discussing a definition, theorem or proof, "it does have to be written on the blackboard because it has to be precise, notation has to be set up, things have to be checked." Informal content usually lacks this degree of detail, and indeed can sometimes be difficult to portray accurately via a written medium. M3 explained this with her comment that "if you write something informal [students] can often misinterpret what you've said and write something entirely different in their notes, so it's a bit problematic, it's a bit tricky to convey this extra information." In addition to the board being useful for expressing precise content, some interviewees noted that some instructors may view the blackboard as being reserved for precise content. When discussing why informal content is usually only delivered orally, M6 related that "I know of people who want to be extremely precise…and that's why they will only write the things that are absolute certainties."

Writing on the Board Slows the Instructor Down Three of the interviewees remarked that a virtue of writing things down on the blackboard is that it slows the pace of instruction and gives the students a chance to comprehend what is being taught to them. Especially when it comes to formal content, the time it takes to utter a definition, theorem, or proof may not be enough for a student to properly analyze it. Writing these things down in addition to speaking them allows students extra time to process them. M7 exemplified this notion with his comment that "the proofs also should be written down…otherwise it would just be too fast to follow."

Writing Formal Content Emphasizes the Language, Notation, and Nature of Mathematics Two mathematicians gave responses that illustrated this theme. M4 mentioned that writing formal content down on the board helps students understand a key fact about the nature of the subject matter, "the idea that mathematics consists of definitions, theorems, and proofs." M7 also noted that mathematics requires a commonly established language and notation, and that writing formal content down helps to achieve this classroom goal. The goal here for these two participants is that students understand the general nature of how formal mathematics is expressed which (naturally) requires seeing the expression of formal mathematics.

Not Writing Informal Content on the Blackboard

Oral Presentation is Needed to Hold Students' Attention Five participants contributed to this theme, which contrasts the level of engagement students have with written versus oral content. The mathematicians noted that sharing informal content is often a matter of telling students how you would like them to be thinking and reasoning. This is best achieved orally, and not through writing. Along these lines M1 stated

> When you're facing the students and you're talking to the students they're more engaged. So when you're trying to explain something that's not formal but you're trying to give an idea of what's going on and how they should be thinking about it, then you want to have them engaged.

M3 expressed a similar opinion that oral, non-written content has the ability to awaken students from a seeming daze.

> Well, in my experience when the instructor maybe puts down the chalk and turns to the front of the class and addresses the class with some anecdote or some informal way of thinking about a concept…that's when students start paying attention.

Written content, on the other hand, can lead to students mindlessly copying down what they see without giving it thorough consideration. Speaking about written content, M5 worried that "they're just transferring it onto each of their notebooks, and whether they actually are getting anything out of it it's not clear." One goal expressed here is that students should be engaged during their advanced mathematical lectures; the stilted process of writing points down can diminish this engagement. This point is interesting as we imagine many mathematics educators would question the assumption that speaking to students more informally would be sufficient to obtain meaningful engagement.

Informal Content is Conversational in Nature Three interviewees gave responses that suggested informal content is itself conversational, and therefore is more naturally conveyed to students in an oral and non-written fashion. M1 described informal content as a "flow of ideas" and said that "if I'm trying to have a discussion with somebody about how you think about this informally, then writing it down on the board converts a discussion into a stilted process." M6 further characterized informal content as a conversation when he stated that "the very nature of informal discussion is that it's not precise." M2 noted the full power of informal content must be exchanged orally, saying "I don't think you're going to inspire people as to the importance of the big ideas without giving a verbal…description of those big ideas." Here the participants are expressing the importance of informal content, but felt that oral presentation is the best way to present this content, which often lacks the precision of formal mathematics.

Content Should Be Repeated to Be Noticed Two participants contributed to this final theme. M3 noted that "research shows that in any room for any presentation, regardless of the topic, at one moment in time, at any given moment in time, one third of the people are not actually listening. And so…to get across an important point, you have to repeat it three times." M2 hypothesized that the big ideas that are usually contained within informal content are repeated enough for students to take notice of them, whereas formal content that is not often repeated can use the blackboard to garner students' awareness. He shared that "I feel like I have to write those technical ones down far more often and the broader ideas get repeated just continually so that I don't."

Discussion

The results we have presented support our claim that instructors of advanced mathematics have valuable observations to make regarding the regularities that mathematics education researchers find in their research. While education researchers can be said to have given a descriptive account of the teaching of mathematicians, discussion with mathematicians can reveal the beliefs and goals that account for the prevalence of the practices we see.

The perspectives that mathematicians bring to math education research have a considerable practical implication. If we wish to change how advanced math classes are taught, it is imperative that we understand how instructors view the practices that comprise their instructional activity. For example, if mathematicians view a particular practice as undesirable and are unaware that it is common among instructors, then to change it may simply require bringing it to the attention of the larger teaching community. If mathematicians see a practice as undesirable but are aware of its widespread use, then this suggests that they may need assistance in the development of teaching practices that can take its place. However, if mathematicians express reasons for viewing a practice as desirable despite math education researchers' aversion to it, then to change such a practice it may be most fruitful to explore alternative practices that possess the qualities that mathematicians find useful and favorable about the current one.

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