Finding Free Variables as a Conceptual Tool in Linear Algebra.

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*This preliminary report examines students' interpretations of free variables in linear algebra. In linear algebra, students build understandings of concepts, such as a (in)consistent system of equations, a linearly independent set of vectors, and a subspace. All these concepts will be the foundation for students' future learning in various fields. Therefore, it is crucial to investigate the notion of free variables as it is one of the constructs underlying work with each of these concepts. Here, I analyzed 110 linear algebra students' written assessments from three different classes using grounded theory (Strauss & Corbin, 1994). The analysis shows that students use free variables as a conceptual tool to answer questions given in different problem settings. This paper reports categories of students' interpretations of free variables and explores what the free variables mean to students in each category.*

*Keywords:* Free Variable, Column Space, Linear (In)dependence, Consistent System

## Literature Review and Theoretical Framework

In linear algebra, students reason and compute with a set of vectors in a matrix form. Thus, how to interpret a set of vectors has a large effect on students' learning in linear algebra. Larson and Zandieh (2013) found students have three interpretations to the matrix equation of **Ax=b**, where A=$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, x=$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, b=$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$; (a) Linear combination interpretation of $x_1a_1+x_2a_2=b$ giving weights $x_1$ and $x_2$ to the column vectors of $a_1$ and $a_2$ being equal to the resultant vector b. (b) System of equations interpretation $a_{11}x_1+a_{12}x_2=b_1$ and $a_{21}x_1+a_{22}x_2=b_2$ viewing the entries of A as coefficients to the linear equations and entries of x as a solution set to the same system of equations. (c) Transformation interpretation T: x→b, where T(x)= Ax, reaching the vector b by multiplying A to the input vector x. Larson and Zandieh (2013) offer evidence about how students interpret the matrix multiplication of Ax=b and how students view the matrix A as well. Students may interpret the matrix A as a collection of column vectors or a collection of row vectors or neither of them. Also, Larson (2010) found that students have two different computational strategies for performing matrix multiplication; linear combination column vectors and row-focused computation. In this sense, I adopt Larson and Zandieh (2013) and Larson (2010)'s perspective for interpreting the matrix equation as my theoretical framework.

Possani (2010) pointed out three types of students' difficulties interpreting with the matrix equation Rx=b′ shown in Figure 1, where R is a row reduced matrix obtained by applying elementary row operations to the matrix equation Ax=b. The first type of students is not able to unfold the form Rx=b′ into the corresponding system of equation. The second type of students plugs a few numbers into the free variables but does not know what to do with them. The third type of students finds just a particular solution by substituting numbers for free variables. The three types of students' difficulties raise the issue of how students treat free variables when they appear in a row reduced matrix.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} -200 \\ 100 \\ 200 \\ 200 \\ -100 \\ 0 \end{pmatrix}.$$

*Figure 1. Rx=b′ obtained from Ax=b (Possani, 2010)*

Harel (2017) mentions how limited in-service teachers' conceptions of free variables can be. This issue came out of the discussion about the relation between a solution, $\mathbf{x}=\boldsymbol{\alpha}+t\boldsymbol{\beta}$, of a non-homogeneous system $S_1$ and the solution, $t\boldsymbol{\beta}$, of its associated homogeneous system $S_2$, where t is a free variable, and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are column vectors. Harel urged the in-service teachers to substitute 0 or 1 into the free variable so that they could find out the relation that the solution for a homogeneous system is nothing but a special case of the solution for the non-homogeneous system. However, the in-service teachers were reluctant to substitute 0 or 1 for the free variables $t_1,\ldots t_k$, wherein the solution for the non-homogeneous is given in a vector form of $\mathbf{x}=\boldsymbol{\alpha}+t_1\boldsymbol{\beta_1}+\ldots+t_k\boldsymbol{\beta_k}$ because they conceived that putting specific numbers in free variables is mathematically illegitimate. The in-service teachers did not allow the free variables to range freely across many values, leading me to question, "what in nature do free variables mean to students?"

Dogan (2018) found that students connect the absence of identity matrix in RREF (Row Reduced Echelon Form) with the existence of a non-trivial type of solutions. In other words, students conceive that the existence of free variables guarantees the existence of non-trivial solutions to the system. Wawro (2014) investigated students' quotes on free variables in her study of students' reasoning about the invertible matrix theorem. A student mentioned that "If RREF is linearly dependent, you're going to have a free variable, then there has to be more than one input to get the same output" This student conceives that linear dependence guarantees a free variable(s) and the existence of free variables allows for multiple inputs to reach a certain value. The studies of Wawro (2014) and Dogan (2017) provide evidence of students' understanding that the existence of free variables is closely related to concepts in linear algebra, such as linear dependence and one-to-one transformation. Additionally, Hannah et al. (2016) pointed out that the number of free variables is also related to the dimension of the column space and row space in students' conceptions.

Many researchers in linear algebra have investigated students' thinking and understanding of various concepts such as linear (in)dependence, span, and eigen theory; however, students' conceptions of the notion of free variable itself have not been investigated much. Even though students regularly use them, free variables have never been a focus of study. Therefore there is a need to examine students' foundational understanding of free variables and how this understanding affects their further reasoning processes. This study investigates what free variables mean to students by asking the following specific questions:

(1) How do students determine if there is free variable?
(2) What are the roles of free variables when students are solving problems in linear algebra?

## Methodology

The population of this study is one-semester course linear algebra students enrolled in a large research university in the United States. The data comes from three different classes' written assessments; 34 students with Exam A, 37 students with Exam B, 39 students with Exam C. The exams cover linear algebra concepts, such as solving linear systems, matrices, determinants, vector spaces, bases, linear transformations, eigenvectors, and decompositions. The exams consist of pairs of the multiple-choice question and its follow up open-ended question asking why it is chosen, T/F question and its follow up open-ended question justifying the answer, and independent open-ended questions. Students are required to show their work for each question to receive full credit. All the students' work was digitally scanned before getting graded and documented in the alphabetical order by last names and then shared in Dropbox folder.

This study analyzed the data using the technique of grounded theory (Strauss & Corbin, 1994). At the first stage, what students mentioned about free variables was explored in the context of the problem provided by the full version of the written assessments. As performing the initial open coding based on constant comparative analysis, the first level of categories emerged; (a) row-centered free variable interpretation **R** and (b) column-centered free variable interpretation **C**. The way students view the location of a pivot in RREF determines whether it is **R** or **C**. Focusing on the two interpretations of **R** and **C**, the second level of categories emerged; (a) linear independence **LI** and linear dependence **LD**, (b) column space **CSP**, and (c) consistent system of equations **CS** and inconsistent system of equations **IS**. These three categories disclose how students conceive free variables in relation to other concepts, such as linear independence, column space, and consistency. Students' answers on every one of the concepts with free variables were coded accordingly.

## Result

In this section, I report findings on the notion of free variables represented in students' work by the categories that emerged during the two phases of data analysis.

**How to determine if there is a free variable(s)**

**1. Row-centered interpretation "R"**; this category connects the existence/lack of pivot in rows with the existence of a free variable. Students in **R** affirm the existence of free variables when there is a lack of the pivot in any "row". Once any row of zeros is found, students recognize the lack of pivot for the row, taking a free variable from that row. Figure 2 illustrates students' work that focuses on the relationship between the pivot row in a row reduced matrix and the existence of the free variable. The students marked boxes of ones representing them as pivots and interpreted that there exists a free variable $x_4$ due to the lack of the pivot in the last row since the last row consists of all zeros. These are classified into **R** in that students obtain the free variable from the row-centered interpretation. This is consistent with Larson and Zandieh (2013) and Larson (2010)'s perspective in that students' notion of free variables varies with row-centered views on the vectors that make up a matrix.

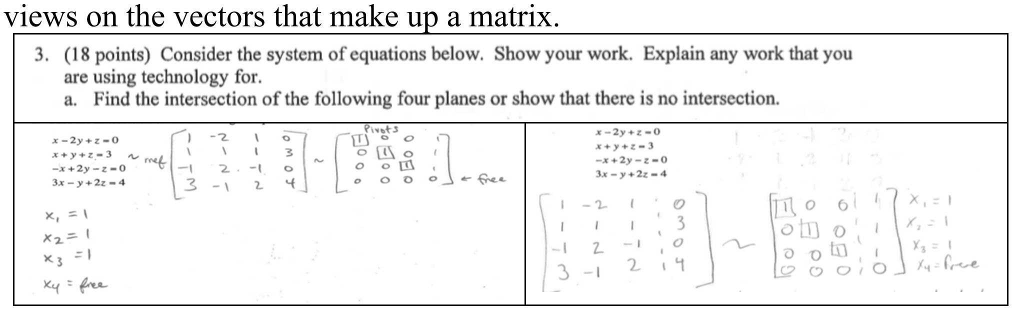
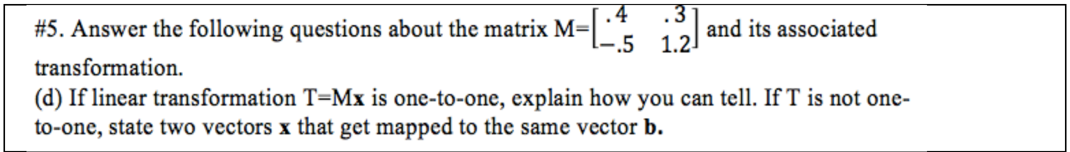
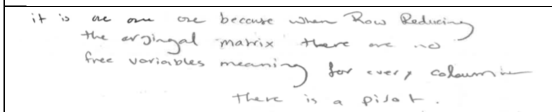


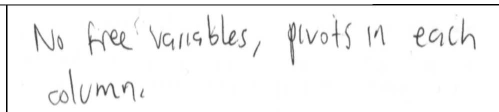

*Figure 2. #3(a) of the assessment and students' answer (Exam A); "**R**"*

**2. Column-centered interpretation "C"**; this category connects the existence/lack of pivot in columns with the existence of a free variable. Students in C affirm the existence of the free variable when there is a lack of pivot in any "column". Once the lack of pivot for any column is found, students take a free variable(s) from that column. Figure 3 illustrates students' answers that focus on the relationship between the pivot column in a row reduced matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ obtained from $\begin{bmatrix} .4 & .3 \\ -.5 & 1.2 \end{bmatrix}$ and the existence of free variable(s). These are classified into C in that students identify free variables from the column-centered interpretation. This is consistent with Larson and Zandieh (2013) and Larson (2010)'s perspective in that students' notion of free variables varies with column-centered views on the vectors that make up a matrix.


#5. Answer the following questions about the matrix M= $\begin{bmatrix} .4 & .3 \\ -.5 & 1.2 \end{bmatrix}$ and its associated transformation.
(d) If linear transformation T=M**x** is one-to-one, explain how you can tell. If T is not one-to-one, state two vectors **x** that get mapped to the same vector **b.**



" it is one to one because row reducing the original matrix there are no free variables meaning for every 'column' there is a pivot."

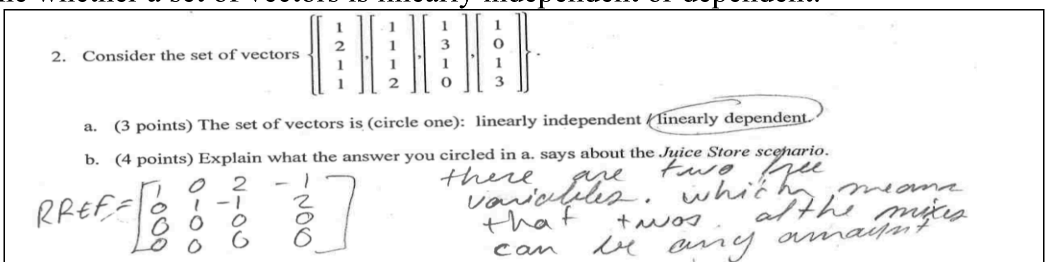


" No free variables, pivots in each 'column' "

*Figure 3. #5(d) of the assessment and student's answers (Exam B);* ***C***

**What to do with the free variables**

**1. Linear independence "LI" vs. linear dependence "LD"**; this category shows students' use of the existence of free variables with the concepts of linear independence/dependence. Figure 4 illustrates a student's solution that focuses on the relationship between linearly dependence and the existence of free variables. The student finds free variables as a sufficient way to confirm linear dependence. Finding free variables is a tool to determine whether a set of vectors is linearly independent or dependent.


2. Consider the set of vectors $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$.

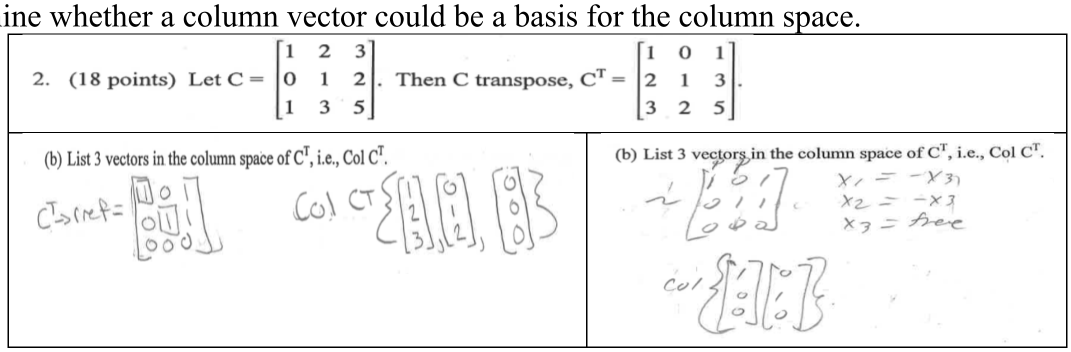
a. (3 points) The set of vectors is (circle one): linearly independent / linearly dependent

b. (4 points) Explain what the answer you circled in a. says about the *Juice Store scenario.*

" there are two free variables. which means that twos of the mixes can be any amount "

*Figure 4. #2(a), (b) of the assessment and a student's answer (Exam C);* ***LD***

**2. Column Space "CSP"**; this category shows students' use of the existence of free variables to find bases of column space. Figure 5 illustrates students' solutions focusing on the relationship between basis vectors in column space and the existence of free variables. The students interpret that finding columns with a pivot(s) is sufficient for the columns to be the basis vectors in the column space. Students selected the first two columns to be the vectors in the column space since those columns have pivots within it. Finding free variables is a tool to determine whether a column vector could be a basis for the column space.

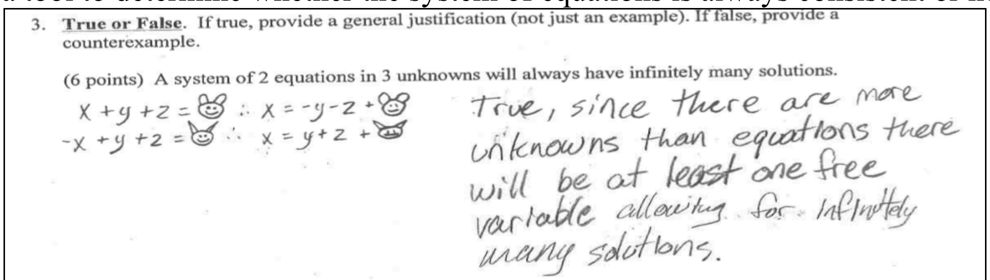

2. (18 points) Let C = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 3 & 5 \end{bmatrix}$. Then C transpose, $C^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{bmatrix}$.

(b) List 3 vectors in the column space of $C^T$, i.e., Col $C^T$.

(b) List 3 vectors in the column space of $C^T$, i.e., Col $C^T$.

*Figure 5. #2(b) of the assessment and students' answers (Exam B):* ***CSP***

**3. Consistent System of equations "CS"**; this category shows students' use of the existence of free variables to determine whether it is a consistent system of equations. Figure 6 illustrates a student's answer focusing on the relationship between the number of solutions of a system of equations and the existence of free variables. The student identifies the existence of free variable(s) from more unknowns than the number of equations and concludes that the free variable allows the system of equations to have infinitely many solutions. Finding free variables is used as a tool to determine whether the system of equations is always consistent or not.


3. **True or False**. If true, provide a general justification (not just an example). If false, provide a counterexample.

(6 points) A system of 2 equations in 3 unknowns will always have infinitely many solutions.

True, since there are more unknowns than equations there will be at least one free variable allowing for infinitely many solutions.

"~ more unknowns than equations… at least one free variable allowing for infinitely many solutions"

*Figure 6. #3 of the assessment and student's answer (Exam C);* ***CS***

In the second level of categories emerged as LI/LD, CSP, and CS along with the existence of free variables, students have different perspectives to view a matrix; as a collection of columns or as a collection of rows (Larson & Zandieh, 2013; Larson, 2010).

## Discussion

I came up with two levels of categories on free variables found in students' written assessments analysis. Figure 7 shows the categories disclosed throughout the different problem settings.

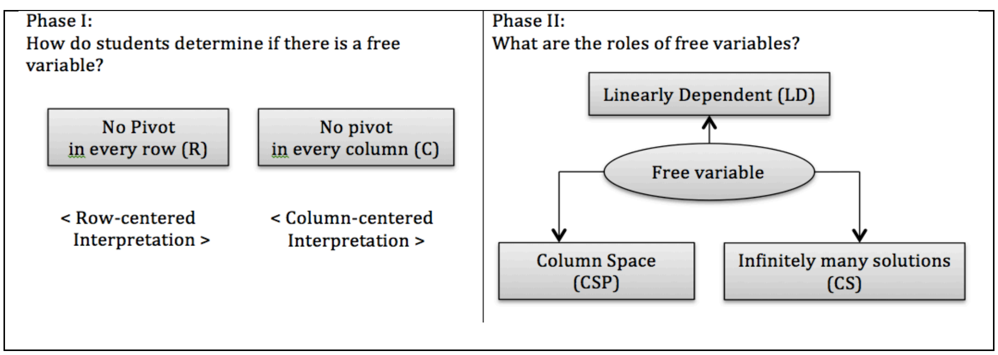



*Figure 7. Categories with two phases of students' meanings of free variables*

Students mention the term 'pivot' frequently with the term 'free variables', however, I could not determine how students actually define 'pivot' from this written assessment analysis. Nevertheless, I could say that students note the existence of free variables from where the lack of a pivot in RREF appears. In addition to finding free variables, students link free variables to other concepts. In other words, students utilize free variables as a conceptual tool since free variables are used to answer questions related to the concepts, such as linear independence/dependence, basis vectors in column space, and consistent/inconsistent system of equations. Due to the nature of analyzing the comprehensive written assessment, it was not quite easy to discern how students justify the connections between the existence of free variables and other concepts. Despite the limitations, this study discloses many issues in students' learning on a set of vectors of a matrix in linear algebra.

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