Example Spaces for Functions: Variations on a Theme

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In this study we focus on example spaces for the concept of a function provided by prospective secondary school teachers in an undergraduate program. This is examined via responses to a scripting task – a task in which participants are presented with the beginning of a dialogue between a teacher and students, and are asked to write a script in which this dialogue is extended. The examples for functions fulfilling certain constraints provide a lens for examining the participants' concept images of a function and the associated range of permissible change. The analysis extends previous research findings by providing refinement of students' ideas related to functions and the concept of the function domain.

Keywords: function, script writing, example space

The Function Concept

The concept of a function is fundamental in mathematics, and it has been repeatedly regarded in the education literature as a central concept in the mathematics curriculum from school to undergraduate studies (e.g., Ayalon, Watson, & Lerman, 2017; Dreyfus & Eisenberg, 1983; Dubinsky & Wilson, 2013; Hitt, 1998; Paz & Leron, 2009). However, it has been demonstrated that undergraduate students often struggle with similar difficulties as those attributed to secondary school students. These include difficulties in recognizing what is or is not a function, especially in cases of "irregular" curves; difficulties in defining what a function is, and not alluding to the definition when working with functions; difficulties in linking and changing between different representations of functions; incorrect assumptions that all functions are continuous and smooth, or need to be expressed as a single formula, equation, or rule; overemphasis on graphic representation and reasoning (such as the vertical line test); and overreliance on procedural algebraic computations (compiled from Dreyfus & Eisenberg, 1983; Even, 1998; Hitt, 1998; Leinhardt, Zaslavsky, & Stein, 1990; Sánchez & Llinares, 2003; Steele, Hillen, & Smith, 2013; Thomas, 2003). As articulated by Huang & Kulm (2012), these types of mistakes are "serious and striking" (p. 427).

The current study is focused on function examples generated by a group of prospective secondary school teachers in an undergraduate program in response to an imagined mathematics-classroom situation. We analyze the generated examples, and demonstrate how the collective example space of the group provides insight into students' ideas and conceptions of a function.

Theoretical Underpinnings: Example Spaces and their Features

Watson and Mason (2005) introduced the notion of *example spaces*, which are collections of examples that are central in mathematical teaching and learning, in the sense that they "require the learner to see the general through the particular, to generalize, to experience the particular as exemplary to appreciate a technical term, theorem, proof, or proof structure, and so on" (p. 4). Example spaces include not only exemplifying mathematical objects, but also a range of related associations and construction methods (Goldenberg & Mason, 2008). Subsequently, Watson and Mason (2005) borrowed and extended terminology from Marton and Booth's (1997) Variation Theory to describe the structure of example spaces. They used the term *dimensions of possible variation* to address the generality of example spaces, meaning those example characteristics that may be varied without changing their exemplifying essence. Additionally, with the associated term *range of permissible change*, they referred to the defining "borders" of example spaces, meaning the extent to which each dimension may be varied. As explained by Goldenberg and Mason (2008), the latter term was introduced to address learners' "unnecessarily restricted sense of the scope of change available in any given dimension" (p. 187). Furthermore, Sinclair, Watson, Zazkis, and Mason (2011) described the following features of example spaces: *population*, meaning how scarce or dense available examples are within an example space; *connectedness*, that is whether different examples in a space are interconnected; *generality*, namely whether the example represents a class of related examples; and *generativity*, which regards "the possibility of generating new examples within the space using given examples and their associated construction tools" (p. 301).

Within the discussion on example spaces, special attention has been given to learners' capability of generating new examples in order to enlarge their example spaces and deepen their understanding of the related underlying mathematical structures. Accordingly, it has been argued that *learner generated examples* (LGEs) can be used as a valuable pedagogical tool to promote conceptual learning and understanding (Watson & Mason, 2005; Watson & Shipman, 2008). Zazkis and Leikin (2007) extended this argument, noting that LGEs are a valuable research tool, since the generated examples provide researchers with a lens into learners' cognitive structures.

The Study

The Participants, Course, and Scripting Task

The participants of the study were twenty prospective secondary school teachers who were studying in a teacher-education undergraduate program. At the time of data collection they were in their final term, enrolled in a course titled "Investigations in Mathematics". During the course the participants completed a series of scripting tasks, one of which is described below and serves as the data for our report.

The task that was presented to the participants of the study belongs to the genre of scripting tasks. In such tasks, participants are typically given a beginning of a dialogue between a teacher and students, referred to as a prompt, and are asked to extend the dialogue in a way they find mathematically fit. Scripting tasks were used in prior research in various mathematical contexts (e.g., Zazkis & Kontorovich, 2016; Zazkis & Herbst, 2018), and their advantages were elaborated upon in detail (e.g., Zazkis, 2018). In particular, a significant feature of scripting tasks is that they provide students the educational opportunity to consider or revisit the mathematical ideas related to the task, and offer researchers a lens on the script-writers' understanding of these particular mathematical concepts and relations.

The current study focuses on a particular prompt for a scripting task, presented in Figure 1. In addition to writing a script that extends the dialogue (Part-A), the students were asked to explain their choice of the presented instructional approach (Part-B). Furthermore, the participants were asked to note if their personal understanding of the mathematics involved in the task differed from what they chose to include in the scripted conversation with students (Part-C), providing us with a finer-tuned lens into their personal mathematical ideas. In the task the participants were presented with a table of values, and invited to explore an imaginary student question, whether there are functions other than $y = 3x$ that satisfy the same table of values.

From a mathematical perspective, the task was designed to address known misconceptions regarding the function concept that are attributed in the literature to undergraduate students. In particular, the task attends to the phenomenon of linear functions as "overpowering" prototypical examples (e.g., Dreyfus & Eisenberg, 1983) and the reported lack of understanding of the arbitrary nature of how a function may be defined (e.g., Even, 1990). From a pedagogical perspective, the design of the task draws on underlying principles of Variation Theory (Marton & Booth, 1997; Runesson, 2005), which regards variation as a pivotal role in the learning process, as it promotes and facilitates the learner's capability to discern and separate critical aspects of mathematical objects. Accordingly, effective task design should foreground variation against invariance of other aspects in the task (Watson & Mason, 2006). The current task sets the four points in the table of values as invariant "pillars", whilst promoting variation through the exploration of the range of permissible change in which functions satisfying the table of values exist. As claimed by Watson and Mason (2005), through the awareness of the dimension of possible variation, learners' example spaces may be enriched.

Teacher: Consider the following table of values. What function can this describe?
Alex: $y = 3x$
Teacher: And why do you say so?
Alex: Because you see numbers on the right are 3 times numbers on the left
Jamie: I agree with Alex, but is this the only way?
Teacher: …

x	y
1	3
2	6
3	9
4	12
5	
6	

Figure 1: A prompt for the Table of Values scripting task

The participants' responses to the "Table of Values" scripting task comprise the data corpus for this study. The scripts were analyzed with a focus on the particular examples of functions considered in the dialogues. The following research question guided the analysis: *What are the participants' example spaces for a function that satisfies the task? More specifically, what are the dimensions of possible variation and associated range of permissible change that are evident in the collective example space of the participating prospective teachers?*

Analysis and Results

The analysis is presented by the main themes that were identified in the scripts. Both authors independently categorized the different examples included in the scripts, and subsequently resolved any discrepancies by discussion and reconsideration of the identified themes. The structures of the exhibited example spaces were then examined in terms of their population, connectedness, generality, and generativity. We distinguished between examples used in Part-A, that could have been purposefully restricted in the scripts based on pedagogical and instructional considerations, and the examples mentioned in Part-B or Part-C, which pointed to participants' personal example spaces triggered by the task.

In designing the prompt, Jamie's question "is this the only way?" was intended to direct the script-writers to consider and explore alternative functions. Indeed, 11 out of 20 scripts included a variety of examples of other functions that satisfy the given table of values, which we categorized into five different dimensions of possible variation. Figure 2 indicates the frequency of occurrences of each cluster of examples pointing to a common dimension. Note that the overall number of occurrences (21) is higher than their associated number of scripts (11), as in most of these scripts multiple types of examples were considered. However, 9 out of the 20 participants did not produce any alternative functions, other than representational variations on the linear option $y = 3x$. Due to the limited scope of this paper, in the subsequent sections we focus only the first three dimensions of possible variation in the script-writers' example spaces.

Alternative options to $y = 3x$	11 scripts	
Single formula expressions	5	Total: 21
Restricting the domain	5	
Graphical representation	5	
Piecewise functions	4	
Recursive relationship	2	
No production of functions other than $y = 3x$	9 scripts	
Different algebraic representations of $y = 3x$	4	
"Shield"	5	

Figure 2: Dimensions of possible variation in the generated examples

Single formula expressions. Five scripts included single formula expressions to describe functions other than $y = 3x$ that satisfy the given table of values. These included two possible options: the absolute value function $y = |3x|$ (three scripts) and a polynomial function (three scripts). We note that both these function types are continuous functions that are defined for all real numbers. Due to the mathematical challenge involved in generating a polynomial function that satisfies the given table of values, we focus our attention on this option, as illustrated in the excerpt from Logan's script:

Teacher: Well in all of these cases we have assumed something subtle. If we filled the table of values what would we get for the remaining y entries?

Alex: 15 and 18

Teacher: Does it have to be those values? What if I put 16 and 23?

Jamie: … Can you do that?

Teacher: Why not? The points could be modeling anything! There is nothing there that says it has to be a line.

Jamie: Can we find an equation for that though?

Teacher: Certainly, but I need to talk about degrees of freedom. In our table of values we could make up 6 values of y and therefore we have 6 degrees of freedom. Simple enough?

Jamie: Mhmm.

Teacher: So we need to find a polynomial with at least 6 degrees of freedom to describe it, that is a polynomial with at least 6 terms.

Alex: So a 5th order polynomial?

Teacher: Exactly Alex, we could find a polynomial of the form $y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ that fits the table of values.

Jamie: But how can we ever assume that any patterns we see in a table of values continues?

Teacher: An excellent question, short answer is we don't. When we make these equations we are assuming that the trend we observe will continue. When making this assumption we need to look for reasons to explain the trend and then ask if we expect those factors to stay the same. Maybe the data was showing the population of a species but at $x = 5$ more food is introduced or a predator is removed and the species can grow at a faster rate.

While general solutions are usually considered in mathematics as more valuable than specific ones, Zazkis and Leikin (2008) noted that often general examples point to an individual's inability to generate a specific one. In this case, the presented example of a polynomial function can be seen as a generality of Logan's personal example space, while it may also point to Logan's difficulty in producing an explicit formula for the polynomial.

While Logan noted the existence of a polynomial function, Corey provided the polynomial $y = x^4 - 10x^3 + 35x^2 - 47x + 24$ "out of the blue", and left it for the imaginary students in the script to verify that it is consistent with the entries in the table of values. In his commentary, Corey added that the polynomial was generated by a computer program, using matrices to solve systems of linear equations. He felt, however, that this material was inappropriate for secondary school students, and in Part-B he wrote: "The level of math needed to determine the final function is beyond what I consider high school level math. After being given the function the answer can be easily revealed, but it still is not easy." We note that Part-C of the task did not demonstrate any alternative higher-level mathematical explanations on how to find fitting polynomials.

Restricting the domain. Five scripts included an example of the function $y = 3x$ in which the domain was restricted to either integers or natural numbers, as demonstrated in the following excerpt from Jill's script:

Teacher: You plotted the points in the table of values, totally correct. Then you connected the dots using a straight line, what is the assumption here?

Alex: Assumption? ……

Teacher: The table of values only gives you the natural numbers, 1, 2, 3, and so on.

Alex: Oh, I guess I assumed that all the points in between follow the same pattern.

Jamie: Well, I guess so too. But now that the teacher mentioned it, maybe the points in between don't have to follow the same pattern?

Alex: I guess so… because they are not in the table of values anyways.

Teacher: That's right! So what other functions can you have?

[Alex and Jamie look at the graph and think.]

Alex: Can we just have those points in the table of values?

Jamie: Like this?

Alex: Yah. It looks a little wired. But it is still a function, right?

Jamie: Right, because it passes the vertical test. It is a function. How do we write the equations then?

[Alex and Jamie feel stuck here.]

Teacher: What is the difference between graph 1 and graph 2?

Jamie: Graph 1 has all the x values, and graph 2 only has natural numbers.

Teacher: Can you describe this difference in more mathematical terms?

Alex: They have different domains?

Teacher: Right, now, can you write the domains for both functions?

Alex: The first one is all real numbers.

Jamie: The second one is all natural numbers.

Teacher: Exactly, when you write the equations, you need to specify domains. By restricting the domains, you have different functions.

As opposed to the previous section, the function examples here are neither continuous nor defined for all real numbers, yet the domain consists of an infinite and unbounded set of numbers. Moreover, these examples demonstrate a recognized human tendency of "continuing the pattern" (e.g., Rivera, 2013), that is, assigning the same rule of multiplication by 3 to all integers. In this sense, the assignment of the same rule to a restricted domain demonstrates the arbitrary choice of the domain in the function concept, though not the arbitrary choice of correspondence between the domain and codomain. In terms of the features of example spaces, on the one hand we note the connectedness between the examples, highlighted through the different attributes of non-identical domains. On the other hand, we notice a "missed opportunity" for generativity, as these examples do not lead to additional generated examples in the scripts that allude to the various options for choosing the domain.

Graphical representations. While in the above excerpt from Jill's script, the teacher confronts students' tendency to connect the points, in other scripts "connecting the points" appears to be the convention that is either supported or invited by the teacher. Taylor exemplifies this tendency:

Teacher: Excellent question Jamie, what's your instinct, are there other ways?
Jamie: Well I don't know, I guess there could be, but how could we tell?
Teacher: Why don't we start by plotting these points. And by we I mean you.
[Students plot the points]
Teacher: Good, so how would it look if we used Alex's function?
Jamie: It would have a straight line through all the points.
Teacher: Yes, but how else can we connect these points?
Jamie: I suppose we could do a zig zag line.
Teacher: Sure, that would work. But we want this to be a function, so what rule do we need to follow?
Jamie: The vertical line test.
Teacher: Which is the easy way of remembering what?
Jamie: Each output can only have 1 input.
Teacher: Correct, so how can we connect these points then?
Jamie: Any way we want as long as we don't break the vertical line test.

In this excerpt, the teacher's question "how else can we connect these points?" leads students to explore alternative options to the straight line. All other scripts that used graphical representation as dimension of variation also alluded to the arbitrary choice of how to "fill the gap" in between the points, presented both via verbal explanations and graphical illustrations, including also non-continuous "step functions" (see Figure 3 taken from one of the scripts). All examples in this dimension explicitly or implicitly regarded the domain as the set of all real numbers. As in Taylor's script above, in the other scripts the determining factor for how to "connect" the points was the vertical line test, serving as the identifying criterion for a function.

While in this dimension, connecting the points extends the population feature of the example space, various ways of connecting the points "anyway we want" (in student words) indicate the generativity, as well as the generality, of the resulting example spaces. However, and in line with previous arguments, this generality may be accompanied by the participants' inability to produce specific algebraic representations for the graphically represented examples.

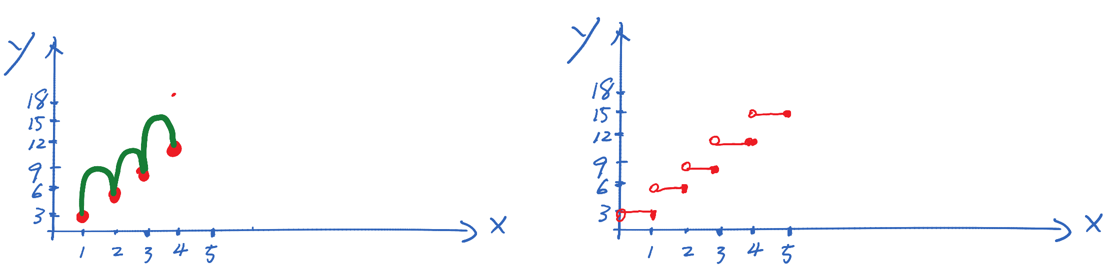

Figure 3: Graphically represented functions

Discussion

The scripts in response to the "Table of Values" task provided a lens into the participants' personal example spaces of functions. Whereas in almost half of the scripts the example space was limited (i.e., no production of functions other than $y = 3x$), the other scripts demonstrated example spaces that were well connected. Within these, the population feature of the participants' example spaces was not extensive; however, generality and generativity were featured in scripts that included multiple examples.

More specifically, the analysis led to two kinds of observations in regard to the participants' example space and concept image of a function (see Vinner, 1983). First, the participants' example spaces provided further support to features that have been previously discussed in the education literature. The students' examples clearly demonstrated the conception that a function should be represented by a single formula (e.g., Vinner & Dreyfus, 1989), typically describing a continuous function (e.g., Hitt, 1998) in which "other points follow the same pattern". Moreover, students' reliance on the "vertical line test" (see Wilson, 1994) was clearly present in the scripts as an identifying criterion for a function.

Secondly, the participants' example choices point to a specific identifying feature of undergraduate students' example spaces of functions, which was not elaborated upon in prior research: that the domain of a function is infinite and unbounded. Focusing on the domain, Bubp (2016) noted that in an attempt to prove mathematical statements, students often used "implicit, unwarranted assumption that the domain of the function f was $\mathbb{R}$" (p. 592) and that "a function cannot have a restricted domain" (p. 593). The current findings provide further refinement of this issue, by noting that even in the examples in which the domain was in fact restricted, it still included infinitely many points (integers or natural numbers). We note that no example of a finite domain or a function on a bounded interval was given by any of the students.

Viewing the findings in a broader context, we suggest that the analysis of scripting tasks not only can provide a theoretical contribution for research, but also a practical utility for undergraduate instruction. In the current case, we also used the analysis of the scripts to plan for follow-up activities that were based on the collective example space of the group, with the goal of extending the participants' personal example spaces, and in such extending their understanding of the concept of a function. These activities are elaborated in detail in Zazkis and Marmur (2018). However, as an illustrative example, one of these activities focused on generating an explicit formula for a non-linear polynomial function consistent with the given table of values. During this activity, we provided one of the examples from the students' scripts, a polynomial of degree 3, for classroom discussion. This led to recalling the Fundamental Theorem of Algebra, and to the subsequent realization that the example was not feasible, as there cannot be a cubic function that intersects a line in 4 different points. This discussion, which made an explicit connection between undergraduate and secondary school mathematics, highlighted the "borders" of the relevant example space, or in Watson and Mason's (2005) terms, the range of permissible change. To conclude, the seemingly simple task of considering a given pattern in a table of values – an exercise that often appears in middle school mathematics lessons – served to advance mathematical understanding of undergraduate students. This by utilizing the collective example space found in the scripts as a springboard for describing the structure of this space, examining what kind of functions belong to the space, determining its confining borders, and enriching the examples of functions that exist within it.

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