Student Reasoning about Eigenvectors and Eigenvalues from a Resources Perspective

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Eigentheory is an important concept for modeling quantum mechanical systems. The focus of the research presented is physics students' reasoning about eigenvectors and eigenvalues as they transition from linear algebra into quantum mechanics. Interviews were conducted at the beginning of the semester with 17 students at two different universities' during the first week of a quantum mechanics course. Interview responses were analyzed using a Resources (Hammer, 2000) framework, which allowed us to characterize nuances in how students understand various aspects of an eigentheory problem. We share three subthemes of results to illustrate this: interpreting the equations graphically, interpreting the equals sign, and determining solutions.

Key words: Linear algebra, Eigentheory, Resources, Physics, Student Understanding

In 2012, the National Research Council's DBER report stated, "The United States faces a great imperative to improve undergraduate science and engineering education" and advocated for more interdisciplinary studies to explore "crosscutting concepts ... and structural or conceptual similarities that underlie discipline-specific problems" (p. 202). In *Project LinAl-P* (NSF-DUE 1452889) we pursue research in this vein by investigating how students reason about and symbolize eigentheory in linear algebra and in quantum physics. For this paper, we explore the following research question: What ways of reasoning about eigenvectors and eigenvalues of real 2x2 matrices exist for physics students at the beginning of a quantum mechanics course?

Literature Review

Research on students' understanding of eigentheory has grown over the past decade, and it provides several insights into the complexity of the topic, students' sophisticated ways of reasoning, and pedagogical suggestions for overcoming the challenges students face. Thomas and Stewart (2011) noted students' difficulty with and need to understand both how matrix multiplication and scalar multiplication on the two sides of the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ yield the same result and how inserting the identity matrix is necessary when symbolically transforming $A\boldsymbol{x} = \lambda\boldsymbol{x}$ into the homogeneous equation $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$. They also advocate for instructors to help their students develop a graphical conception of eigenvectors and eigenvalues, something they noted was weak in their study participants. Gol Tabaghi and Sinclair (2013) investigated students' visual and kinesthetic understanding of eigenvector and eigenvalue. The authors analyzed the results in terms of Sierpinska's (2000) modes of reasoning, finding that students' work with the sketch and their interaction with the interviewer promoted the students' flexibility between the synthetic-geometric and the analytic-arithmetic modes of reasoning.

Henderson, Rasmussen, Sweeney, Wawro, and Zandieh (2010) illustrated, prior to any instruction on eigentheory, various ways that students interpreted $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$ (see Figure 1 part (a)). The authors parsed students' activity through their symbol sense, noting if they conducted superficial algebraic cancellation to conclude that $A = 2$ or if they interpreted the equals sign as a signifier of balanced results. The authors also found that of the students that were about to find the solution given a specific *A,* only some were able to interpret their results. This may relate to Harel's (2000) suggestion that the interpretation of "solution" in a matrix equation, the set of all vectors $\boldsymbol{x}$ that make the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ true, entails a new level of complexity than does solving equations such as $cx = d$ (where c and d are real numbers). Finally, in physics education research, Dreyfus, Elby, Gupta, and Sohr (2017) examined students' attempt to reconstruct the time-independent Schrödinger equation. The researchers focused on the relationship between the symbolic forms for eigentheory and the various meanings they could imbue for students, noting "parsing the conceptual meaning of mathematical expressions and equations can play a key role in mathematical sense-making" (p. 11). These particular aspects – the meaning of symbols and the objects they represent, graphical interpretations, and interpreting solutions – are all particularly relevant for our present study and help inform our analysis.

Theoretical Framework

To operationalize the research question, we assume a theoretical stance consistent with what Elby (2000) calls "fine-grained constructivism" in which "much of students' intuitive knowledge consists of loosely connected, often inarticulate minigeneralizations and other knowledge elements, the activation of which depends heavily on context" (p. 481). This is consistent with the Knowledge in Pieces theoretical framework (diSessa, 1993), which utilizes an assumption that students' intuitively held knowledge pieces are productive in some context. To conduct research on student understanding consistent with this theory, we characterize students' cognitive resources (Hammer, 2000) that are utilized when they engage in activity related to eigentheory in quantum physics. Sabella and Redish (2007) defined a resource as "a basic cognitive network that represents an element of student knowledge or a set of knowledge elements that the student tends to consistently activate together" (p. 1018). Resources are activated depending on how individuals frame a given situation, that is, how an individual unconsciously interprets what is happening around them (Hammer, Elby, Scherr, & Redish, 2005). Individuals may sometimes have the resources needed to solve a given problem but fail to activate them, activating instead other less-productive resources. However, all "resources are useful in some contexts, or they would not exist as resources" (Redish & Vicentini, 2004). Resources can be linked to other resources, in which activation of one resource can promote or demote activation of others. Furthermore, resources may internally consist of finer-grained resources linked in a particular structure (Hammer et al., 2005; Sayre & Wittmann, 2008). In our research, we seek to identify resources that characterize the knowledge elements quantum physics students activated when reasoning about eigenvectors and eigenvalues of a real 2x2 matrix.

Methods

The data consist of video, transcript, and written work from individual, semi-structured interviews (Bernard, 1988), drawn on a voluntary basis, with 17 students enrolled in a quantum mechanics course. Nine were from a junior-level course at a large public research university in the northwest US (school A), and eight were in a senior-level course at a medium public research university in the northeast US (school C). Student pseudonyms are "A#" or "C#." Interviews occurred during the first week of the course, and questions aimed to elicit student understanding of several linear algebra concepts which they would use in the quantum mechanics course.

For this paper, we focus on students' reasoning on one particular interview question. There were additional follow-ups to check that the interviewer understood the students' points, but below the five main prompts to the question are in Figure 1. Parts (a)-(c) were introduced in Henderson et al. (2010) in their research on student thinking prior to any formal instruction on eigentheory. Because linear algebra was a prerequisite for the quantum mechanics courses in which our participants were enrolled, we knew they would have been exposed to eigentheory prior to the interview. By design, the terms "eigenvector" and "eigenvalue" did not appear until part (e); many students, however, immediately recognized the equation in (a) and brought up eigentheory ideas on their own in their responses to (a)-(e).

(a) Consider a 2×2 matrix A and a vector $\begin{bmatrix} x \\ y \end{bmatrix}$. How do you think about $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$?
(b) Do you have a geometric or graphical way to think about this equation?
(c) How do you think about what the equals sign means when you see it written in the context of this equation?
(d) Now suppose that $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$. Now how do you think about $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$? What values of x and y would make the equation true?
(e) [If they hadn't already] Again consider the matrix A = $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$. Determine the eigenvalues and eigenvectors of A.

Figure 1. The main interview question prompts for the analyzed data.

Analysis was done through an iterative process of individual coding, group discussion, and codebook development. First, three members of the research team individually coded (Miles, Huberman, & Saldaña, 2014) transcripts of the student interviews, specifically assigning codes for what each researcher felt represented evidence of a student's resources that were activated as they answered the eigentheory questions. Next, researchers discussed their individual codes noting the specific evidence within the transcript used to mark that code. These codes and the evidence that identify them were extensively discussed, refined and solidified. Based on these discussions, a coding book was developed with labels and descriptions of the agreed-upon resources; this three-step process was repeated until the coding book was sufficient to characterize the thinking of all seventeen students. Finally, we individually coded each transcript one more time, achieving a high level of interrater reliability using the final codebook.

Results

In total, our analysis of student reasoning about eigentheory in this interview led to the identification of over 50 resources. When considered in small subsets or in total, these resources allow us to characterize nuances in how students understand various aspects of the concepts involved in an eigentheory problem. We share three subthemes of results to illustrate this: interpreting the equations graphically, interpreting the equals sign, and determining solutions.

Interpreting the Equations Graphically

Student responses to part (b) demonstrated a wide variety of ideas that are shown in Figure 2. Common among them were ideas about a matrix acting on one of its associated eigenvectors and scaling the eigenvector, while others stated that the matrix acts on one of its associated eigenvectors and stretches the eigenvector. Some students made comments about eigenvectors being "Anything on this line" referring to the line of the eigenspace defined by the eigenvector. Some students described that an eigenvector when acted on by its associated matrix provides a resultant vector on the same line.

Intriguingly, when students were first asked about geometric and graphical interpretations in part (b), nine students drew vectors on a plane and discussed the meaning of the eigenvector, the associated matrix, and eigenvalue; however, after giving students an explicit 2x2 matrix in part (d), five additional students engaged in this activity; we coded this as activating the *Vector Graphing* resource. Ultimately, 14 of 16 students that were asked this question activated a resource that connects the idea of an eigenvector, eigenvalue, and associated matrix with a 2-D plot of vectors. Having a specific example of a two-by-two matrix, and determining that a solution to the system of equations is also a solution to the eigenequations (which 14 students were able to do) seems to trigger this graphical drawing resource for an additional five students. Although most students activate geometric/graphical ways of thinking about the eigenequation, different students require different support and feedback to activate this idea.

As an example, responding to part (b), C3 said, “Not really because...I know. No. I wouldn't really say so. [*after another prompt from the interviewer*] Well I think about an eigenvector as this being a vector that when multiplied by something stays along the same path.” The student states that they don’t have any graphical or geometric way of thinking about this problem but eventually states an idea that we would code as Evec-Line.

After completing part (e) the interviewer asks: “Ok. Umm. Now that you have like actual numbers for A or numbers for that the vectors do you have any additional graphical or geometric ways you think about it?”

C3: “Umm... No. Not really. I would just say [draws two coordinate axes] that u 1 would look -- I really wouldn't, I wouldn't really think about it like this but I would say that u 1 looks something like this [draws vector into fourth quadrant] and u 2 looks something like this [draw another vector into second quadrant collinear with first vector].

Despite the insistence that the student “wouldn’t really think about it like this,” the student provides a clear vector graph consistent with the eigenvectors for the matrix.

Resource	*Resource Description*	*# activating resource*
Vector Graphing	Student talks about vectors as arrows on a Cartesian plane, or actually draws a graph with vectors or an “eigenline” on it.	14
Val-Stretch	Mentions that in an equation of the form A[x;y]=k[x;y], [x;y] is stretched by k. This captures any of the more geometric ideas like dilation, longer, etc.	8
Val-Scale	Mentions that in an equation of the form A[x;y]=k[x;y], [x;y] becomes k times that vector or is scaled by k. This captures any of the more algebraic ideas.	10
Evec-Line	Student explains some version of that an eigenvector of *A* lies along the same line or goes in the same direction after being acted on by *A*	6
Evec-Eigenspace	Student explains that vectors "in the same direction as" or "on the same plane as" or "the same line" as other eigenvectors of *A* would also be eigenvectors.	5

Figure 2. List of resources most related to graphical interpretations of the eigenequation.

Interpreting the Equals Sign

The seven main resources that were activated in response to part (c) are listed and defined in Figure 3. Although these resources were grounded in and grew from our data, our familiarity with the literature allowed us to notice when our students’ reasoning was consistent with a way of reasoning already documented in the literature. For instance, *RU*= and *OU*= are used to characterize student responses that seem to stem from either a relational or operational understanding of the equal sign. The terms “operational” and “relational” were used by Knuth, Stephens, McNeil, and Alibali (2006) to categorize students’ explanations of what the equal sign means (see also Behr, Erlwanger, & Nichols, 1980; Carpenter, Franke, & Levi, 1999; Kieren, 1981); those with an operational understanding view the equal sign as a signal to “compute” or “give the answer,” while those with a relational understanding view the equal sign as indicating a relation between the two sides of the equation, with one side being “the same as” the other. An example of *OU*= from our data is C5’s statement: “I think about it in terms of eigenvalue, I'm saying that with this matrix there is some eigenvalue that solves, that there is some unique value that corresponds to matrix *A* that solves this equation.” An example of *RU*= is below with C7.

The resources *Algebraic Cancellation* and *SOSE* are closely related. The former was introduced in Henderson et al. (2010), who used this term to describe overgeneralizing the notion of algebraic simplification to “cancel” the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ from both sides of the equation in part (a) and then trying to make sense of how the matrix *A* could equal the number 2. The resource *SOSE* characterizes student efforts to find a way to turn the matrix *A* into the number 2. Finally, the resources *PV-mult* and *OV-mult* are used to characterize student thinking that centrally considers the operations on either side of the equal sign and/or the resulting objects. The resource *PV-mult* indicates a student response fixated on matrix and scalar multiplication being different processes, whereas *OV-mult* indicates a student response highlighting that the result of matrix and scalar multiplication is the same object. We chose these resource names as a reference to the work by Thomas and Stewart (2011) who used the term “process-object obstacle” to describe “how the two sides of the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ represent different mathematical processes that have to be encapsulated to give equivalent mathematical objects” (p. 280).

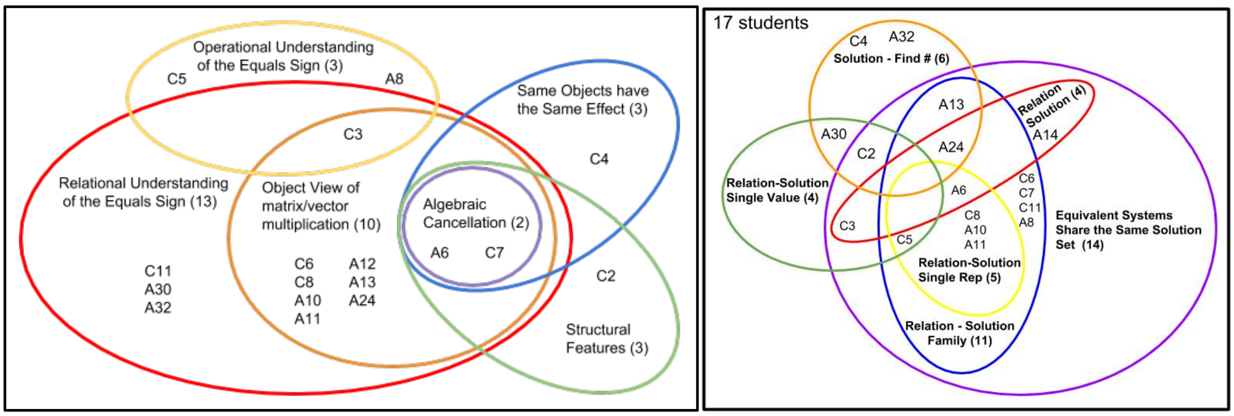
Resource	*Resource Description*
RU =	Relational Understanding of Equal Sign means that entities on both sides of the equation must be "the same"
OU =	Operational Understanding of Equal Sign is a call to "do something" such as solve an equation or "compute."
SOSE	[Things have to be the Same Object to have the Same Effect] For the equation to make sense, there has to be a way to turn the matrix A into the number 2.
Structural Features	Student discusses the structure that the objects in the equations have. This often entails discussing or comparing one or more of the entities in the equation.
Algebraic Cancellation	If the same thing is on both sides of an equation in a structurally similar way, it is permissible "cancel" those things out of the equation.
PV - mult	[Process view of matrix and scalar multiplication] Student focuses on matrix multiplication being different from scalar multiplication (the student focus is on the operation).
OV - mult	[Object view" of matrix and scalar multiplication] Result of matrix multiplication and scalar multiplication is the same object (the student focus is on the objects created by the operation).

Figure 3. List of resources most related to interpreting the equals sign in $A\boldsymbol{x} = 2\boldsymbol{x}$*.*

Figure 4a illustrates how the various resource codes loaded across the 17 student responses in part (c) (only one student, A6, activated *PV-mult*, so it is not in Figure 2a). We note that this question is often difficult for students; some find it hard to describe their understanding without using the word “equals” (which they were prompted to do if needed), and many seem to be figuring it out as they respond. This latter aspect can be seen in students’ responses such as C7, whose explanation was coded with 5 of the 7 resources in Table 2.

C7: Well it's weird cause it almost seems like *A* equals 2. You know what I mean? Like *A* has to equal to 2 for this to be equivalent, but *A* is not equal to 2. *A* is a matrix. So, that's what- I never thought of that but *A* does not equal 2. *A* is equal to a 2 by 2 matrix [draws brackets for a matrix]…which is not equal to 2 but it's like...The *A* on its, on its own does not equal 2 but the *A* operating on *xy* does equal 2 times *xy*. So, this group together [circles LHS in equation of problem statement] is equal to this group together [circles RHS]…But when you say, 'oh lets, let's divide both sides by *xy* vector' [makes air quotes]. That doesn't make sense linearly, I don't think. But, you- intuitively a lot of the time I guess in algebra- from algebra experience, you'd think *A* matrix is equal to 2.

The first four lines of C7’s response, when he grappled with how to reconcile that A can’t equal 2 even though it seems like it does, was coded with *SOSE* and *Structural Features*. He moved towards resolving this with the statement “the *A* on its, on its own does not equal 2 but the *A* operating on *xy* does equal 2 times *xy*," which was coded with *OV-mult*, and by stating the two groups on either side of the equation were equal, which was coded with *RU=*. His conclusion, which brings up dividing both sides by a vector and how that is sensible in algebra, was coded as *Algebraic Cancellation.* We note that it is most likely the case, based on his activation of *OV-mult* and *RU=*, that C7 was confident that $A \neq 2$; however, we still also code his response with *Algebraic Cancellation* because this resource was activated for C7 during his thought process.



(a) (b)

Figure 4. Venn Diagram of main resources activated by students in Part (c) and in Part (d), respectively.

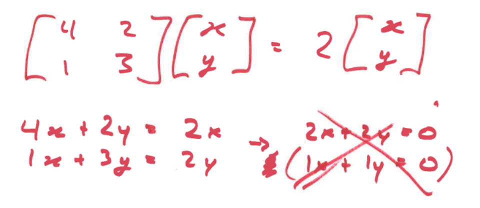
Determining Solutions to the Matrix Equation

In response to part (d), students activated six main resources to make sense of the solutions to the matrix equation, with some resources being more productive than others. We share these in Figure 5 and summarize the resource activation by the 17 students in Figure 4b.

Resource	*Resource Description*
Solution-Finds #	In a system of equations, the solution should be a single number for each variable.
ESS	Algebraically equivalent equations or systems of equations share the same solution set.
Relation-Solution	A relationship can define what values for the unknowns are solutions to the given equation(s).
Relation-Solution Single Rep	A single representative of a relationship can be used as a prototype or to check the solution.
Relation-Solution Single Value	A relationship that is a solution to a system of equations defines a single solution.
Relation-Solution Family	A relationship that is a solution to a system of equations defines an infinite number of possible solutions.

Figure 5. List of main resources for finding solutions to $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$.

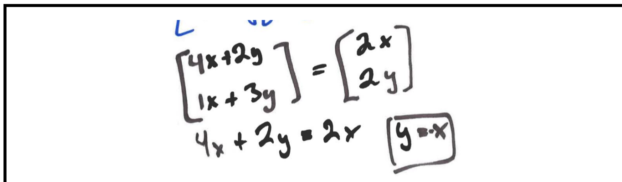
The resource *Solution-Find #* was activated by 6 students, implying they thought that the solution to the equation should result in single, specific numbers for both x and y. This was most often coupled with the students attempting to use the elimination or substitution methods for solving systems of equations. For example, consider C4's work and thoughts in Figure 6. C4 attempted to use the elimination method on the system of equations he had produced from the matrix equation but became stuck as the equations "cancelled" each other. In fact, C4, as well as A32, could not think of any other ways to approach the problem, and both were not able to find any solutions at all to the matrix equation.



"Could multiply that side by 2…no that doesn't work … I was thinking multiply that side, ya know, so that you'd get, so you could subtract one side from the other…But the fact that it's 2x + 2y = 0 for one equation 1x + 1y = 0 doesn't really. Ya know if I multiplied by 2 to cancel one of the variables and then subtract both variables are cancelled [crosses out system]. So obviously so that doesn't work in that in that sense."

Figure 6: C4's attempt to find solutions to $\begin{bmatrix}4 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 2\begin{bmatrix}x\\y\end{bmatrix}$.

In contrast, consider C3 who eventually realized that the equations define a relationship between x and y, which we coded as an activation of *Relation-Solution*: "Hmmm -- wait I think from here I can say that ... no ... What if I said -- So I could 2*y* equals minus 2*x* [writes 2*y* = -2*x*]. So now we're getting somewhere." While C3 did eventually realize the importance of this relationship, C3 was also one of the four students who activated the *Relation - Solution - Single Value* resource, thinking there should still only be one solution to the matrix equation, determined by the relationship. Another student who activated the *Relation - Solution - Single Value* resource, C2, recognized that he was trying to find an eigenvector, explained that eigenvectors must be normalized, and attempted to find this normalized vector. When the interviewer asked, "Is that the only one for $\lambda = 2$?" C2 replied, "Yes." We note this might be evidence that the *Relation - Solution - Single Value* resource could stem from students' nascent knowledge that quantum mechanical states (including eigenstates) are represented by normalized vectors due to the probabilistic nature of quantum mechanics.



C11: "Yeah. And since it's only *y* and *x*, I can just plug in any value of *y* and *x* that satisfies this equation [draws box around *y*= -*x*] and it will satisfy that same one [points to $\begin{bmatrix}4 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 2\begin{bmatrix}x\\y\end{bmatrix}$.]

Figure 7: C11's explanation of solutions to $\begin{bmatrix}4 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 2\begin{bmatrix}x\\y\end{bmatrix}$.

Impressively, 11 of the 17 students eventually concluded that the relationship $y = -x$ or $x = -y$ actually defines an infinite number of solutions, as any values of x and y which satisfy that relationship will be a solution to the matrix equation. For instance, consider C11's response in Figure 7. C11 also exemplifies an important resource that a large majority of the students (14 of the 17) activated as they worked through this problem, namely ESS. As students algebraically manipulated the matrix equation into other forms, it was notable most recognized that solutions to these new equations would also be solutions to the original matrix equation.

Conclusion

In this study, we identified a variety of resources that characterize students' thinking as they reasoned about eigenequations for 2x2 matrices during an interview at the start a course on quantum mechanics. The three themes presented here – reasoning about the equals sign, reasoning geometrically, and reasoning about solutions – represent a subset of the results that were obtained through our analysis. Our aim was to not identify incorrect reasoning but rather understand the various resources that students found useful at some point in the context of the interview question. Our analysis sheds light on both productive and occasionally unproductive resources for understanding eigentheory. These are helpful for instructors and curriculum developers to know so that they can help students build upon the common resources or seek to refine why certain resources aren't appropriate to activate in particular contexts.

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