Investigating STEM Students' Measurement Schemes with a Units Coordination Lens

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Measurement is a foundational concept in all STEM fields. Difficulties with measurement and converting between units of measure have been documented in medical students, chemistry students, and mathematics students at varieties of educational levels. However, less is known about why this topic is so difficult and what mental operations are entailed in mastering it. Steffe (2012) argued that students must assimilate situations with three levels of units to understand measurement conversions so we attend to students' units coordination schemes while remaining open to other factors impacting students' responses to measurement tasks. We found that the STEM majors in our sample who assimilated tasks with two levels of units had more difficulty with measurement tasks than those who assimilated tasks with three levels of units.

Keywords: Measurement, Units Coordination, Quantitative Reasoning, STEM Majors

Research on quantitative reasoning is an important area of Research in Undergraduate Mathematics Education (Thompson, 2012). Thompson (2012) defined quantification as the mental process of conceiving of some aspect of an object as measurable and understanding that the measure of the object is some multiple of the chosen unit of measure. Steffe presented a conceptual analysis of the cognitive foundations of quantitative reasoning and measurement by building on his research into children's coordination of partitions and iterations of multiple units (Steffe, 2013). We use Steffe's units coordination constructs to understand students' thinking about measurement. Given students' difficulties with measurement it is important to understand the conceptual roots of the issues. Thus our research question is:

How are STEM majors' units coordination structures related to their understanding of measurement?

Literature Review

Measurement and conversions that are fundamental in many STEM courses (e.g., DeLorenzo, 1994; Saitta et al., 2011; Scott, 2012). However, there is evidence that these ideas are poorly understood. Large samples of university calculus students and secondary mathematics teachers found it difficult to convert between liters and gallons given a conversion factor (Thompson, Carlson, Byerley & Hatfield, 2013; Byerley & Thompson, 2017; Byerley, 2016). Difficulties with measurement are also common in doctors with medical degrees. For example, in one study there were 55 medication errors per 100 patients admitted with 28% of those errors related to prescribing appropriate doses of medicine (Kaushal et. al., 2001). Chemistry students struggle to interpret what it means to perform dimensional analysis. This has driven many to investigate more effective methods of teaching this technique; for instance, by including descriptive words with calculations (DeLorenzo, 1994), having students work collaboratively with manipulatives (Saitta, Gittings, & Geiger, 2011), and using interactive software that shows the sizes of units (Ellis, 2013). Chemistry educators debate teaching dimensional analysis as rote procedure vs deliberately scaffolded logic and reason (DeLorenzo, 1994).

Less is known about why measurement is so difficult. One hypothesis is that many students do not assimilate situations with three levels of units when they are asked to make sense of measurement in elementary school (Steffe, 2013). Smith and Barrett (2017) conjecture that part of the difficulty might be the way measurement is taught, and the lack of focus on the underlying structures of various measurement situations.

> [We] found it striking how often the same conceptual principles and associated learning challenges appear in the measurement of different quantities… Despite the clear focus in research on equipartitioning, units and their iteration, units and subunits… curricula (and arguably most classroom teaching) focus students' attention on particular quantities and the correct use of tools, as if each was a new topic and challenge (p. 377).

Our study investigates STEM majors' units coordination schemes and their measurement schemes to describe the conceptual structures needed to understand measurement.

Theoretical Perspective

Steffe (2013) posited that students need to be able to assimilate situations with *three levels of units* to make sense of measurement situations where one quantity is measured with more than one unit. He also explained how the ability to assimilate situations with *two levels of units* is related to being able to construct a measure of one quantity. Steffe and colleagues came to these conclusions based on teaching experiments with mostly K-8 students (Steffe & Olive, 2010) and their hypotheses have not been investigated with undergraduate students.

Units Coordinating

Units coordinating is described as "students' ability to create units and maintain their relationships with other units they contain or constitute" (Norton, Boyce, Phillips, Anwyll, Ulrich, & Wilkins, 2015). Units coordinating is foundational for the construction of early whole-number concepts, such as n 1s being equivalent to one n (Steffe & Cobb, 1988). Units coordinating is a useful construct for understanding students' fractions conceptions. To understand the fraction m/n as a number, one must understand m/n as commensurate with m $1/n$ths, n of which are commensurate with 1 (Hackenberg, 2010). In the case of $m > n$, the meaning of $1/n$ must transform from thinking of $1/n$ as one out of n total pieces to thinking of $1/n$ as an amount that could be *iterated* more than n times without changing its relationship with the size of 1 (Steffe & Olive, 2010; Tzur, 1999). This involves coordinating three levels of nested units: 7/3 is 7 times (1/3), 1=3/3 is 3 times (1/3), thus a 7/3 unit contains both a unit of 1 and a unit of 1/3 within 1 (see Figure 1). Students thinking this way about fractions are said to have constructed an *iterative fraction scheme* (Steffe & Olive, 2010).

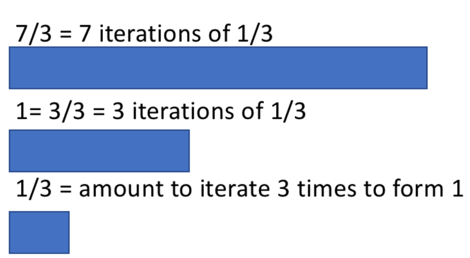

Figure 1. Three level structure for 7/3

Students coordinating with fewer levels of fractional units may construct measurement conceptions of fractions limited to proper fractions (i.e., *partitive* fraction schemes) or be limited to conceptions of fractions disconnected from measurement (i.e., *part-whole* fraction schemes; Steffe & Olive, 2010). Students who can assimilate with two levels of units can often correctly solve tasks that have a three-part unit structure if they are able to use manipulatives or images. We say these students can coordinate three levels of units in activity, but do not assimilate tasks to a three-part structure that they have already constructed mentally. In other words, although the two-level students do not simultaneously keep track of three units and their relationships in their mind they can cope with this three-part structure using tools and correctly solve many problems.

Reciprocal Reasoning

Construction of an iterative fraction scheme is necessary for *reciprocal reasoning,* which has connections to students' reasoning in school algebra (Hackenberg & Lee, 2015) as well as measurement. To construct reciprocal reasoning, students must abstract a structure for their coordination of three levels of fractional units that can apply more generally to *unknown* units (Hackenberg and Lee, 2015, p. 226). For instance, consider the equation $y = 7/3\ x$. A student employing reciprocal reasoning may reverse the multiplicative relationship, to obtain $x = 3/7\ y$, by understanding that each $\frac{1}{3}$ of x is 1/7 of y, so 3/3 of x is 3/7 of y (Hackenberg, 2010). Reciprocal reasoning is one form of reversible multiplicative reasoning - a student may instead reverse a multiplicative relationship by reasoning about reversing whole number arithmetic operations (e.g., by multiplying 3 and dividing by 7). This ostensibly yields the same result, but it is disconnected from the multiplicative relationship between the *x* and *y.*

Methods

We recruited eight calculus students from two universities by visiting calculus courses and asking for volunteers. We interviewed all students who volunteered. Six students were enrolled in Calculus II at one university and two students were enrolled in Calculus I for Biologists at the second university. Each student was interviewed individually for approximately one hour by one of the authors and answered units coordination and measurement tasks. We report on three students whose reasoning illustrates trends we saw in all interviews.

The interview protocol included seven units coordination items developed and validated by Norton et al. (2015) for assessment of middle school students' reasoning. We chose these tasks because there was guidance from prior research on how to use them to diagnose units coordination structures. The most difficult task in the assessment is shown in Figure 2.

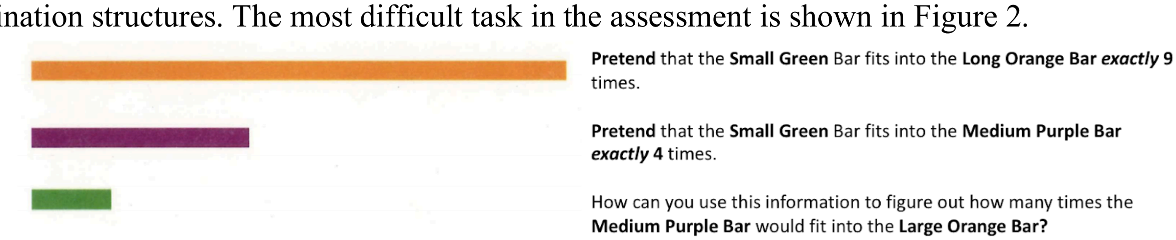

Figure 2. The task "Measuring Bars" from Norton et.al. 2015.

The liters to gallons conversion task was developed for secondary mathematics teachers (Byerley & Thompson, 2017). We knew this task was very difficult to solve correctly based on prior research, but did not know what made the task so challenging for people with math degrees.

A container has a volume of m liters. One gallon is $\frac{189}{50}$ times as large as one liter. What is the container's volume in gallons? Explain.

Figure 3: The task "Liters to Gallons" from Byerley and Thompson, 2017. © Arizona Board of Regents 2015.

The other measurement items came from assessments and worksheets in the first author's Calculus for Biologists course. These included drawing a ruler with both centimeters and inches on it, determining the number of square centimeters in one square inch, and doing unit conversions with fictional units given a conversion factor: A Mump is 7/3 times as large as a Tog. We chose these tasks because we knew they were difficult but did not know why.

Our research team watched video recordings of each interview and made initial notes about how students responded to the units coordination and measurement tasks. After independently making notes summarizing each interview, we individually wrote descriptions of the students' responses to units coordination and measurement tasks. If we all determined a student assimilated tasks with three levels of units independently we felt more confident in our model of that students' thinking. We shared our notes and discussed differences in our interpretations, using the discussion as a chance to identify and test multiple conjectures that could explain the students' activities.

Results

We will compare and contrast our interpretations of three students' units coordination and measurement schemes. Students 1 and 3 were independently categorized by all team members as assimilating situations with three levels of units. Student 2 was categorized by all team members as assimilating situations with two levels of units.

Students' Responses to *Measuring Bars*

The research team used students' responses to seven units coordination tasks to decide how many levels of units the student assimilated with. We will discuss the evidence from the most difficult task "Measuring Bars." It is the most difficult because unlike the other tasks the answer is not a whole number. Table 1 summarizes features of each students' response.

Table 1. Summary of three students' responses to the Measuring Bars Task

Student	Answer or Answers	Time to Giving Correct Answer	Needs an image to produce answer?	Number of levels assimilated?
Student 1	9/4	42 sec	No.	Three
Student 2	2 1/9 then 2 1/4	4 min 15 sec	Yes.	Two
Student 3	9/4	49 sec	No.	Three

Student 1. Student 1 correctly answered all of the tasks on Norton et. al.'s (2015) instrument without needing supporting images, which is evidence he assimilated the situations with three levels of units. The short amount of time he took to solve the *Measuring Bars* Task suggests he was able to assimilate the task to his existing three level unit structure. We also considered other evidence of his use of a three-level unit structure in his strategies for partitioning bars. For example, when partitioning a bar into 6 equally sized pieces (the fourth bar task) the student first partitioned the bar into three equally sized pieces, then partitioned each of

those pieces into two equally sized pieces (he used a similar strategy to make 12 inches on the ruler task: split the ruler in half, each half in half, each quarter into thirds).

Student 2. Student two was able to coordinate three levels of units in activity with the aid of pictures and repeated prompting but did not assimilate tasks with three levels of units. When faced with tasks involving improper fractions, she expressed a preference of converting them to decimals.

Unlike Student 1, Student 2 identified that she could not solve the *Measuring Bars* task (Figure 2) without drawing a picture. Even with the support of the picture she did not keep in mind relationships between three quantities. Student 2 answered "two and one out of nine." She was fairly confident in her answer of 2 1/9 but also considered "two and one out of four" before choosing 2 1/9. The interviewer told her that one green bar is one ninth of an orange bar and asked her what fraction one green is of a purple bar. Student 2 determined correctly the green bar is one fourth of the purple bar but then reconfirmed "so I think my answer should be 2 and one ninth." The conversation continued:

I: How did you decide you should write that fraction in terms of the size of the orange versus the size of a green or a purple?"

S: Like you said, it got me thinking, that makes sense, because this whole one is a green one, and when we look at it in terms of orange it is just one ninth of an orange, the question is asking to answer in terms of the long orange bar so I decided it would be one ninth.

I: Does this to you also refer to long orange bars. [points to the 2 in the answer 2 1/9.]

S: That refers to how many purple fits into the long orange bar. So it would be two purples and an extra of the green. [student laughs]

I: Okay. And the extra green is one fourth of one purple.

S: Oh. [sense of realization]

I: So this answer is correct in the sense that you mean two purples and one…[gets cut off]

S: one ninth of a green.

I: [corrects student] one ninth of an orange.

The conversation continued until Student 2 decided to change her answer to 2 ¼ (the intended answer). Student 2 had difficulty keeping track of three units in her mind as evidenced by calling a green square both one ninth of a green and one ninth of an orange. She also does not remember her measure of two is in terms of the purple unit when she determines the size of the leftover green piece. Steffe hypothesized that constructing an iterative fraction scheme to understand nine fourths requires assimilating the situation with three levels of units. In this case understanding that the green is one fourth of the purple while at the same time thinking of the orange as nine copies of the green.

The interviewer asked the student if the answer of 2 1/4 was related to the nine and four given in the problem statement. She replied:

> Ummm... I think it is related to the nine? [questioning tone.] Ummm... I would usually check my work using like a calculator because I'm not really good with fractions. I don't usually do fractions, I would put it into decimals. So I guess like two point two five would fit into nine... [pause of six seconds to compute.] like four times. So that would be four times two point one four to get the nine.

This excerpt provides evidence that Student 2 does not have an iterative fraction scheme. Student 2 was not aware that 9/4 was the same number as 2 ¼, as indicated by the multiple pauses and computations the student made when asked how 9 and 4 were related to her answer of 2 ¼.

Student 3. This student was able to answer units coordination tasks correctly before drawing any pictures, but sometimes would make units-related errors when discussing his reasoning (e.g. mixing up number of purple and green bars). His response to *Measuring Bars* was distinctly different than Student 2's and demonstrates the student likely had a meaning for division as producing a measure of two quantities and is comfortable with fractions like 9/4. His ability to answer *Measuring Bars* quickly without an image suggests he assimilated the task to an existing three-unit mental structure in his mind. He explained his answer of 9/4:

> Basically, the small green bar into purple is four, uh, the green bar into the full thing is nine, so if I take the full thing and I want to know how many of these there is. I'm basically just using green as units, so it's like the full bar of greens is 9, the purple's size is 4, 9 divided by 4, basically using it as the smallest unit. [Points to the greens on his diagram.] These are the fourths because they are the greens and there are 9 of them.

Student 3's language describing the orange as nine copies of the fourths is consistent with an iterative fraction scheme which students typically construct after assimilating tasks with three levels of units (Steffe & Olive, 2010).

Students' Responses to Measurement Tasks.

Student 1. Student 1 had the strongest measurement schemes in the group of eight students interviewed this summer. He told the interviewer he had not previously seen many of the measurement questions but was able to figure them out fluently without help from the interviewer. For example, Student 1 did not know that there are 2.54 cm in one inch, but given that information by the interviewer he was able to draw an essentially flawless representation of a ruler using a straightedge. He attended to making sure that the 2 inch mark was lined up with 5.08 cm mark and that the 12 inch mark was lined up with the 30.48. He was the only student of eight to correctly answer the *Liters to Gallons* conversion task, which is known to be hard for secondary math teachers (Byerley & Thompson, 2017). He utilized reciprocal reasoning to express x liters as 50/189 x gallons, but keeping track of the distinction between the number of liters (x) and the size of a liter (some agreed upon amount of volume) was non-trivial for him. He reread the prompt four times to make sense of it and spent a few minutes contemplating his answer before feeling confident.

Student 2. Student 2 expressed apprehension about drawing a ruler with centimeters and inches on it despite having memorized that 2.54 centimeters equals one inch. She first drew a picture of a ruler with inches on it. Unlike the students in the interviews who assimilated tasks with three levels of units, she did not partition a partition to form twelve equal parts, and none of her inches ended up the same size. The following excerpt shows that despite the interviewer's attempts to orientate her and help her understand the question she did not come up with a plan for drawing the centimeter side of the ruler.

S: So that would be 2.54.[marks 2.54 cm across ruler from 1 inch]. I don't know. I don't know how like proportionate it should be....

I: By proportionate do you mean that the lines don't usually line up?

S: Yeah. But the big lines do, but like there is like small lines in between that don't. I don't really know what the cm side should look like as I never really use that side. We use the conversion but don't use a ruler to look at it.

I: If you were trying to fit… I'm going to draw it bigger so it is easier to look at….if this is one inch and this is two inches...and then you were trying to put centimeters over here would you be able to? And I agree at one inch you get 2.54 cm, but usually on rulers what they do is that they put whole number values of centimeters. They put whole number values they don't put decimals. Does that make sense?

S: Yeah. I don't think I'll be able to draw the centimeter side in whole numbers, I just know the conversion.

On the *Liters to Gallons* task Student 2 knew that there should be more liters than gallons in a given container but struggled to respond to the question for a variety of reasons. One of her repeated difficulties was choosing between a meaning of x as a number of liters and a meaning of x as the size of one liter. It might be that differentiating between the number of copies of a unit and the size of a unit while also attending to a second unit of measure requires assimilating the task with three level of units. She did not express awareness of the reciprocal relationship between the relative size of units of measure and the measurement of a container. Thus she knew there should be fewer gallons in the container, but did not know how to find the number of gallons precisely. This is consistent with prior observations that assimilating situations with three levels of units is important for development of reciprocal reasoning.

Student 3. Student 3 drew a ruler correctly, and his ruler drawing and other work showed that he understood that if a quantity was measured with a larger unit of measurement, the resulting measure was smaller. However, Student 3 answered the *Liters to Gallons* task incorrectly with the expression (189/50) x. He did not distinguish between the number of liters (x) and the size of a liter in his response. Based on his answers to other questions he seemed to have the unit structures he needed to answer the task, but he did not consider the meaning of x in his expressions and so did not notice it stood for two different ideas. When asked how many mumps were in a tog, Student 3 correctly answered 3/7, but he questioned himself, stating, "You're asking how many mumps are in the tog, so how many big are in the small. So, it'll be a fraction. I'm saying 3/7." There is evidence of reversible multiplicative reasoning in Student 3's immediately attributing the reciprocal of 7/3 to how many mumps are in a tog. But Student 3's pausing and consideration of generic relative sizes ("how many big are in the small") before settling on 3/7, together with his incorrect response to the *Liters to Gallons* task, suggests his reciprocal reasoning involving unknown quantities was not well-established.

Conclusions

This evidence suggests that development of fraction schemes and units coordination structures described by Steffe and colleagues to model children's reasoning is useful for understanding adults' measurement schemes. As his theory predicted, the calculus students who assimilated situations with two levels of units had not constructed productive measurement schemes. Student 2 had developed many strategies (such as dimensional analysis) for understanding problems without needing to assimilate them with three levels of units. However, some of her strategies, such as converting fractions to decimals, made it more difficult for her to make useful observations about reciprocal relationships and were detrimental to her conceptual understanding of unit conversions. It is much easier to see the reciprocal relationship between 50/189 and 189/50 when the numbers are left as fractions. When student 2 converted all fractions to decimals it often made it much harder for her to generalize important aspects of the problem. The example of Student 3 shows that assimilating tasks with three levels of units is not enough to make sense of *Liters to Gallons* without help. Even students with strong units coordination schemes and strong measurement schemes, like Student 1, may still find *Liters to Gallons* difficult. Across our sample of eight, students' units coordination structures are related to their ability to reason about measurement in non-routine ways. Assimilating tasks with three levels of units appears to be necessary, but not sufficient to understand a variety of measurement tasks.

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