Distance Measurement and Reinventing the General Metric Function

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Real analysis is an important course for both undergraduate and graduate students. Researching the ways students reason about challenging and abstract concepts can inform and improve instruction in real analysis. In this report, I examine two undergraduate students' reinvention of a general metric function. To facilitate this reinvention, I conducted a 15-hour teaching experiment with undergraduate mathematics students that had completed the introductory sequence in real analysis. In this experiment, the students generalized their initial understandings of distance measurements on $\mathbb{R}$ to construct increasingly abstract measures of distance in various metric spaces, including sequence and function spaces. Their generalizing activity culminated in construction of a general metric function through reflected abstraction of operations relevant to distance measurement carried out in previous metric spaces. I explore the students' generalizing activity, as well as the abstractions that supported their generalizing.

Key words: generalization, real analysis, metric spaces, formal mathematics

Introduction and Review of the Literature

Success in real analysis can have substantial implications for undergraduate mathematics students, especially those pursuing graduate degrees. Along with abstract algebra, the majority of mathematics majors must take real analysis in some form as part of a core curriculum. Further, real analysis holds implications for mathematics graduate students as well, as it can be a major component of qualifying exams.

Despite its importance, real analysis is anecdotally difficult for both undergraduate and graduate students. In spite of this, we know very little of its teaching and learning, particularly in advanced settings of real analysis. While real analysis is the setting for various research agendas (e.g. proof, classroom instruction, student affect, understanding of definitions, etc.; c.f Alcock & Weber, 2005; Lew, Fukawa-Connelly, Mejía-Ramos & Weber, 2016; Weber, 2009), we know relatively little about how students understand real analysis topics outside of introductory contexts. While there have been a number of studies exploring how students understand formal limits (e.g., Adiredja, 2013; Cornu, 1991; Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996; Gass, 1992; Roh, 2008; Swinyard, 2011; Swinyard & Larsen, 2012; Tall, 1992; Tall & Vinner, 1981; Williams, 1991), students' understandings of other key concepts in real analysis has generally not been explicitly studied.

Three exceptions to this are works by Wasserman and Weber (2018), Strand (2017), and Reed (2017). Strand used the Intermediate Value Theorem in the context of approximating an irrational root to draw out students' understanding of completeness on $\mathbb{R}$. Reed (2017) detailed a case study wherein a student reversed the roles of ϵ and N in point-wise convergence of functions as a result of a similar reversal in his understanding of real number convergence. Finally, Wasserman and Weber (2018) explored ways to use issues of classroom pedagogy in motivating underlying structure in introductory real analysis taught specifically to preservice teachers. While each of these studies explore different facets of student thinking in various real analysis contexts, there is still much we don't know about how students understand core and unifying concepts in real analysis.

In this report, I extend the literature by exploring how students understand more abstract concepts in real analysis, specifically in the context of metric spaces. A metric space, (X, ρ), consists of a set, X, paired with a measure of distance (i.e. a metric function), ρ. A function ρ is a metric if it satisfies the following four conditions for all $x, y, z \in X$: 1) $\rho : X \times X \to [0, \infty)$, 2) $\rho(x, y) = \rho(y, x)$, 3) $\rho(x, y) = 0$ iff $x = y$, and 4) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$. A productive understanding of metric spaces attends to the regularity of their topological structure across spaces in which such a pairing exists. For instance, sequences obey the same convergence structure, in that a sequence $\{x_n\}$ in a metric space converges to an element $x \in X$ if $\forall \epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, we have $\rho(x_n, x) < \epsilon$. Thus, a sequence of continuous functions under the supremum metric converges in the same way that a sequence of real numbers does under the absolute value metric.

I extend our knowledge of student thinking and learning in real analysis by exploring what advanced understandings students can construct through generalization of their understandings developed in introductory real analysis. Specifically, I report on the results of a 15-hour teaching experiment (Steffe & Thompson, 2000) that involved students' reinvention (Freudenthal, 1991) of a general metric space in real analysis.

Theoretical Perspectives

To reinvent the general metric function, the students engaged in generalizing activity that facilitated reflected abstraction (Piaget, 1975, 1980, 2001; Glasersfeld, 1995) of operations (referring to mental actions) involved in measuring distance in physical space. To describe both students' generalizing activity and the cognitive processes that support their learning while generalizing, I draw both from Ellis, Lockwood, Tillema, and Moore's (2017) Relating-Forming-Extending (R-F-E) generalizing framework and Piaget's (1975, 1980, 2001) notion of reflected abstraction. These two theoretical constructs have a synergistic relationship, in that Ellis et al. (2017) offer a nuanced analysis of the ways that students engage in the activity of generalizing, while Piaget's (1975, 1980, 2001) notion of reflected abstraction provides a language describing the underlying cognitive processes behind the students' learning through generalization.

Ellis et al. (2017) identify *relating* and *extending* as two broad categories of actions that students can take while generalizing. Relating occurs when " students [establish] relations of similarity across problems or contexts" (Ellis, et al., 2017, p. 680). This is a form of inter-contextual generalizing, in which students create relationships between two or more mathematical situations that they initially perceive as distinct. *Extending*, perhaps the more recognizable generalizing action, involves the application of established patterns, regularities, and relationships to new cases (Ellis, et al., 2017, p. 680).

Piaget's construct of reflective abstraction (specifically reflected abstraction; c.f. Piaget, 1975, 1980, 2001; Glasersfeld, 1995) complements the attention to the activities in which students engage as they generalize. Through reflective abstraction, we can make inferences about the cognitive mechanisms driving the students' generalizations, as well as the ways in which their knowledge is transformed through generalization. Situated as a mechanism of accommodation that facilitates equilibration (Glasersfeld, 1995), *reflective abstraction* is primarily characterized by two inseparable features: 1) a *réfléchissement* "... in the sense of the projection of something borrowed from a preceding level onto a higher one" (Piaget, 1975, p. 41), and 2) a *réflexion* "... in the sense of a (more or less conscious) cognitive reconstruction or reorganization of what has been transferred" (Paiget, 1975, p. 41). In this way, reflective abstraction captures the ways thinkers regulate their activity by first borrowing operations (mental actions) from one level of mental complexity (say, some Nth level of projected activity) and then reorganizing the operations on a new projected level (i.e. the $N+1$st level). This reorganization produces new mental constructions enriched by the projected operations. Piaget acknowledged when this reconstruction occurs through explicit reflection on a thinker's activity by calling such conscious reflections *reflected* abstraction (Piaget, 2001; Glasersfeld, 1995).

These constructs frame the generalizing actions of the students in my study as they constructed the general metric function by first *relating* across their previous metrics, and then *extending* meaningful structures they identified through the process of relating. This involved explicit reflection on prior operations they had enacted in specific metric contexts that related to the properties of a metric, and so their extending activity occurred through reflected abstraction. I will examine their specific abstractions in the Results Section.

Methods

The data presented here was taken from a dissertation project involving two separate teaching experiments (Steffe & Thompson, 2000) with mathematics majors. Both teaching experiments entailed 8, 90-minute sessions in which students reinvented (Freudenthal, 1991) the general definition of a metric function. In this report, I focus on the final session of one teaching experiment involving a pair of students, Christina and Jerry. Both Christina and Jerry were mathematics majors (Jerry also studied physics while Christina was pursuing teaching credentials) that had completed the introductory real analysis sequence at their university. This two-term sequence covered topological results on the real line under the absolute value metric, as well as a rigorous treatment of basic calculus results including differentiation, integration, and point-wise and uniform convergence of function sequences. Importantly, the students had no previous exposure to metric spaces, or any form of measuring distance other than with the absolute value metric or the Euclidean measure of distance in real space. Thus, their activity with distances in more abstract contexts (e.g. the taxicab and supremum metrics in real space, sequence spaces, and function spaces) was truly novel for them. Figure 1 gives an overview of the specific spaces the students discovered during the latter sessions in the teaching experiment, as well as the major topics of discussion in each space.

$$\overset{\textit{Convergent Sequences}\qquad\textit{Distance Measurement}}{\underbrace{Sequences\ of\ real\ numbers}_{l_1,\, l_2\ and\ l_\infty}} \rightarrow \overset{\textit{Completeness}}{\underbrace{C([a,b])}_{L_1,\, L_2\ and\ L_\infty}} \rightarrow \overset{\textit{Openness}}{(X,\rho)}$$

Figure 1: Overall progression from sequence spaces to the general metric.

Throughout the teaching experiment, the students were given the goal-oriented prompt of characterizing sequential convergence in each new space they explored. This activity necessarily involved the construction of a distance measurement as well. The researcher then guided the resulting student activity primarily by facilitating moments of perturbation, as is consistent with teaching experiment methodology (Steffe & Thompson, 2000) and the RME heuristic of guided reinvention (Freudenthal, 1991).

The specific session that I am reporting on was the last session of the teaching experiment, wherein the students engaged in relating (Ellis, et al., 2017) by reflecting on commonalities across structures they perceived in the distances they had constructed throughout the previous sessions. This reflection on prior activity culminated in the students formally writing out the properties of a general distance function.

Video records were made of each session, and the video records were analyzed using the data analysis software MAXQDA. Specifically, each record was reviewed for moments of generalization, as well as moments of mathematical activity or discourse from which inferences could be made about the students' schemes and accommodations made to their schemes in the process of equilibration. Such instances were then coded according to the R-F-E framework (Ellis, et al., 2017), and also analyzed according to Piaget's constructs. The episodes were then further analyzed through Thompson's (2008) method of conceptual analysis (specifically Thompson's second use of conceptual analysis), consistent with Steffe and Thompson's (2000) concept of model building.

Results

For the purposes of this report, I will give an overview of Christina and Jerry's generalizing activity, and provide a representative sample of episodes that demonstrate their generalizing and abstraction. Recall that a function, ρ, is a metric if it satisfies the following four conditions: 1) $\rho : X \times X \to [0, \infty)$, 2) $\rho(x, y) = \rho(y, x)$, 3) $\rho(x, y) = 0$ iff $x = y$, and 4) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$. I will focus on Christina and Jerry's construction of properties 1-3, as property 4 (the triangle inequality) was generalized through a qualitatively different goal-oriented activity than those from which they generalized properties 1-3. Because of space restrictions, I will only demonstrate the progression of their abstraction of the third metric property, however I will briefly offer a description of the abstracted operations that contributed to the other properties as well.

The prompt and some initial generality

The session began with the students discussing their perspectives on a list of past distances they had constructed. The list, which I provided for them on the board, included the ℓ_1, ℓ_2, and ℓ_∞ distances (metrics) on $\mathbb{R}^n$ and sequences spaces, as well as the L_1, L_2, and L_∞ distances (metrics) on continuous functions defined on a closed interval. After some initial reflections, I reminded them of an earlier time in the experiment where they had expressed the desire to characterize their distances through a general distance function, A. Asking them to reflect on this activity, Jerry commented that " . . . each of these the process is the same, so let's call this thing a general distance A, and then write everything in terms of that." In this case, Jerry's use of the term 'process' conveyed that they each had the purpose of measuring distance on various mathematical objects, as later evidenced by his reflection that ". . . we found that there are other quantities [other distances] that satisfies the rules that we need [i.e. rules of distance measurement] but that may not necessarily look the same." These statements convey the relating that Jerry had engaged in, making explicit perceived relationships between different measures of distance in that they all behaved similarly, with the differences being the objects they acted on.

Jerry's statements were supported by Christina, who posited the "square root Pythagorean theorem [Euclidean distance] is like how we traditionally think of distance between two objects . . . and then the other ones kind of just go into things that we don't normally think about." The students' comments reflect that each new distance they constructed adhered to some collection of "rules" that distances should follow. Their formal defining of the metric function then came about through thematization at the general level (as it occurs in reflected abstraction[1]) of those rules that they had brought out earlier in the experiment. In this way, the students' formal statements of the properties of A were generalizations that occurred through reflected abstraction of the operations carried out in accordance with the "rules" of distance measurement as understood by the students. I will now give a representative sample of such operations that contributed to the students' defining of the third metric property, $\rho(x, y) = 0$ iff $x = y$.

The meaning of zero distance

Jerry and Christina's understandings of 0 distance were largely motivated by their explorations of sequential convergence in the various spaces they examined throughout the teaching experiment. Integral to the characterization of convergence is the tendency of the sequence approximations to the limit point to "tend towards 0". Early on, Jerry and Christina realized that convergence occurring in the way they intended necessitated a meaningful 0 distance measurement. In particular, if the students constructed initial distance measurements that resulted 0 distance measurements for non-similar mathematical objects, they then altered the form of their measurement to achieve a meaning consistent with what we know to be the general metric.

As an example of this activity, I reference the students' initial construction of the taxicab metric from an early session in the teaching experiment. The students initially used the formula $L = |v_x - w_x + v_y - w_y|$. To facilitate perturbation, I asked the students to measure the L-distance between the vectors $\vec{v} = [2, 1]$ and $\vec{w} = [1, 2]$. Upon calculating a 0 distance, I had the following conversation with the students:

Jerry: It seems weird. I don't like that.

Interviewer: And why?

Jerry: Because if we drew a picture, right? [draws two different vectors, $\vec{v}$ and $\vec{w}$, and a difference vector connecting them] We've got a — this is $[1, 2]$, so here's this vector and the other one is going $[2, 1]$. This vector, the distance seems like that should be a number greater than 0, but here we show that it is 0.

Interviewer: And why do you feel like the number should be something greater than 0?

Christina: Because when we're saying that the — in our lines — this is only looking at the convergence [their statement of $L_n \to 0$]. And we're characterizing that by distance of 0. However, if two vectors are converging upon each other, then they're becoming the same vector essentially, and those two things [the vectors Jerry drew] aren't the same vector but their distance is 0.

This discussion facilitated their altering of L to the standard ℓ_1 (taxicab) metric. This refinement of L in this instance constitutes generalizing through *extending*[2]. Their generalizing activity in this instance demonstrates the "rule" that they imposed on their new L distance, primarily that distance measurements of nonsimilar objects should be nonzero. To this point, Jerry later commented that "It seems like a good thing for our distance to be able to do, 'cause we want to use it to differentiate between vectors. If we can't, then sort of what's the point, I guess?" This conveys that Jerry was conceiving of using distance functions as a means of taking two vectors and obtaining information about the differences between them based on the information

[1]Piaget took thematization to mean "to know [something] consciously and in an easily verbalized form" (Piaget, 2001, p.31)

[2]See *operating* in Ellis et al. (2017).

given by the distance measurement. I next give another example from their sessions exploring functions spaces, where this operation of differentiating non-similar vectors was projected to a higher level of organization, and then conclude with the thematization of these operations at the general level during the last session of the experiment.

When exploring measures of distance between continuous functions defined on a closed interval, the first measure of distance the students initially constructed was similar to the L_1 measure of distance in the form of $d = \int_a^b f(x) - g(x)\, dx$. Facilitating a similar perturbation to that of the taxicab metric, I asked the students to measure the distance between the functions x and $x + sin(x)$ on the interval $[-\pi, \pi]$. Responding to this, the students made the calculations in Figure 2 and had a discussion, in which Jerry made the following comments:

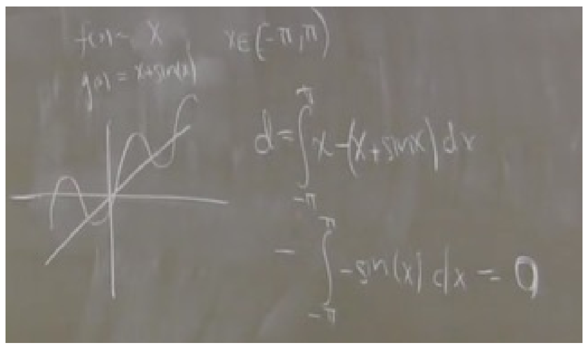

Figure 2: Calculating the 0 distance of x and $x + \sin(x)$

Jerry: . . . we know, visually, that our functions will look like this [draws the graph in Figure 2], and so those definitely aren't the same. . . . Like if I had a sequence of functions that converges to a function, I want to show that they end up becoming the same thing eventually. So it doesn't make sense for these [x and $x + sin(x)$] to have 0 distance but look different. . . .

As before, the students imposed the meaning of 0 distance on their new d function to adhere to some abstracted notion of distance measurement that gave specific meaning to measurements of 0 distance. While sequential convergence was a motivator for this meaning behind 0 distance, ultimately the students generalized by imposing meanings on these function distances generated from an abstracted construct of distance and distance measurement.

This meaning in the general setting was first voiced by Christina. During the final session, after revisiting the above L_1 example above, Christina said that ". . . to say that a distance is 0 means — that like two dissimilar things has distance 0 means that they're on each other essentially, but that means that they are the same thing." Formalizing their understandings, the students then wrote the two conditions $A(u, v) > 0$ if $u \neq v$ and $A(u, v) = 0$ if $u = v$, which simplifies to the third condition of a metric. I infer that this formal statement was a written thematization of the operations that they had projected throughout the teaching experiment related to the activity of differentiating objects through interpreting the result of distance measurement. In terms of their generalizing activity, they engaged first in relating and then in extending (specifically through *removing particulars*[3]), as the specific contexts of the metric spaces were abandoned to reflect the general structure of the distances they wished to convey. Further, as evidenced by their verbal and written thematization, this generalizing activity was reinforced by

[3]See the subcategories of extending in Ellis, et al.'s, R-F-E framework (2017)

reflected abstraction of the operations involved in extracting meaning from a measure of 0 distance between two objects.

The other metric properties

This progression of generalizing and abstracting similarly occurred with each of the metric properties. Metric properties 1) and 2) emerged through attending to operations involved in distance measurement. In particular, the symmetry of the metric function emerged through abstracting operations involved in comparing measurements of reorderings of the objects being measured. Jerry described this property through the analogy of "... the distance between me to the wall is the same as the distance from the wall to me." Further, the first property of a metric emerged through Jerry and Christina both attending to distance as a measurement of "how far apart things are," and that conventional measures of distance primarily convey meaning through nonnegative measurements. This highlights that, for Jerry and Christina, carrying out operations involved in distance measurement (in the sense of conceiving distance measurements physically) was a productive and integral aspect of their generalizing activity to the level of a metric function. Consciously abstracting these operations (as occurs in reflected abstraction) then resulted in their production of the first three general metric properties.

Discussion and Concluding Remarks

This report demonstrates a productive way that students might learn and understand the metric function and its properties, and how the generalizing action of relating can facilitate reflected abstraction. Specifically, as the metric function is a measure of distance, conceiving of the activity of measuring distance, in this case physically measuring distance, can reveal certain operations inherent to distance measurement that can be abstracted to reveal the structural properties of the metric function. In the case of Jerry and Christina, they understood distance measurement as adhering to a certain set of "rules" that they imposed on their various constructed distances throughout the teaching experiment. These "rules" provided them with certain operations (such as the comparing of the similarity between objects in reference to their measurement value or comparing the measurements of the same object pair up to ordering) that they abstracted into increasingly general mathematical settings.

Through engaging in *relating* of their list of specific distances, (i.e. reflecting on the operations through which they constructed the collection of specific distance measurements) the students were able to reflect on and regulate their prior abstractions. In this way, the students engaged in reflected abstraction by "reflecting on reflection"(Glasersfeld, 1995, p. 105) and then *extended* the resulting metric properties to the general level by *removing particulars*. Thus, their generalizing activity was complemented by reflected abstraction of the specific operations that comprised the first three metric properties. Further, the generality of the metric structure was reinforced by the regularity of its occurrence across metric spaces (i.e. various spaces such as $\mathbb{R}^2$ and L_p).

Continuations of this research will investigate the impact of students attending to distance measurement in various abstract spaces prior to introduction of the general metric in a classroom environment. Future research will also explore the role that generalization plays in other constructs vital to real analysis, such as the measure function. This work builds a foundation of the ways that undergraduate students can reason about real analysis when transitioning to a graduate setting.

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