Making Implicit Differentiation Explicit

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This paper discusses the conceptual basis for differentiating an equation, an essential aspect of implicit differentiation. We explain that implicit differentiation is more than merely the procedure of differentiating an equation and carefully provide a conceptual analysis of what is entailed in understanding the legitimacy of this procedure. This conceptual analysis provides a basis for discussion of the literature, as well as empirical justification for the importance of this topic.

Keywords: Implicit Differentiation, Related Rates, Derivative Operator, Calculus

The topic of implicit differentiation has been identified as "missing" from RUME research (Speer & Kung, 2016). This theoretical paper aims to begin to fill that gap by addressing the legitimacy of applying the differential operator to each side of an equation, an essential aspect of implicit differentiation. We take as axiomatic that understanding implicit differentiation involves understanding why it is legitimate to perform the procedure of differentiating each side of an equation.

In this article, we provide a conceptual analysis of what it means for someone to understand the legitimacy of differentiating both sides of an equation. By carefully examining an implicit differentiation problem and a related rates problem, we explain how, despite a procedural similarity, implicit differentiation is not merely the procedure of "taking the derivative of both sides," a conflation that exists even within the math education literature. We discuss literature *after* presenting the conceptual analysis, as the discussion is through the lens of our conceptual analysis. Finally, student data illustrates that understanding the legitimacy of this operation is nontrivial for students.

The Normative Solution to a Ubiquitous Problem

We begin with a pair of ubiquitous problems as well as their standard solutions, which can be found in the implicit differentiation and related rates sections of most calculus curricula. This illustrates how the conceptual basis for implicit differentiation can easily be lost in the implementation of its procedure.

> Suppose a 3-meter ladder, starting flush against the wall, begins sliding down the wall, the top sliding down at 0.1 meters per second.
>
> **(a)** Find the rate of change of the distance of the top of the ladder from the base of the wall with respect to the distance of the bottom of the ladder from the base of the wall.
>
> **(b)** Find the rate of change of the distance of the bottom of the ladder from the base of the wall with respect to time.

Figure 1. The ladder problems

A prototypical solution of problem (a) involves letting x be the distance the bottom of the ladder is from the base of the wall and y be the distance the top of the ladder is from the base of the wall, both in meters. Then we get:

(1a)	$x^2+y^2=9$	From the Pythagorean theorem
(2a)	$2x+2y(dy/dx)=0$	By differentiating with respect to x
(3a)	$dy/dx=(-x/y)$	Solving for dy/dx

This equation yields the relevant rate of change at any point in the ladder's motion. The solution for (b) is similar:

(1b)	$x^2+y^2=9$	From the Pythagorean theorem
(2b)	$2x(dx/dt)+2y(dy/dt)=0$	By differentiating with respect to t
(3b)	$dx/dt=(-y/x)(dy/dt)$	Solving for dx/dt
(4b)	$dx/dt=(0.1y/x)$	Substituting -0.1 m/s for dy/dt

The most commonly used calculus texts (e.g. Rogawski, 2011; Stewart, 2006; Weir, M. D., Hass, J. R., & Thomas, G. B., 2011) present solutions to related rates and implicit differentiation problems similarly to what is above, often without an explanation of the legitimacy of the procedure (Broussoud, 2011). This treatment overlooks why the procedure works and treats the derivative like a basic algebraic operator that can be applied to both sides of an equation (Thurston, 1972, Staden, 1989). However, the derivative cannot simply be applied to both sides of any equation. To illustrate this point, consider the equation x=1. Taking the derivative of both sides of this equations leads to 1=0, an absurdity, whereas applying any basic arithmetic operation yields a related legitimate equation (2x=2, x+2=3, x-1=0, etc.). So clearly there is more to why the derivative of both sides procedure works and when it can be applied than is evident from the prototypical examples above. We explore this in the following section.

A Conceptual Analysis

In order to address why the procedure for solving implicit differentiation problems is valid, we begin with a *conceptual analysis* (under the epistemological perspective of radical constructivism (Thompson, 2008)). This conceptual analysis is intended to put the reader's understanding of the relevant mathematics on solid footing and explicitly lay out conceptual operations involved for one to understand implicit differentiation and differentiating equations robustly. It can thus enhance any framework that addresses problems in which students differentiate equations, such as related rates problems (Engelke, 2007; Martin, 2000). This conceptual analysis facilitates our discussion of student struggles with the validity of the implicit differentiation procedure later in this manuscript.

Let us revisit **(a)** from Figure 1 above. We start as before with the equation $x^2+y^2=9$, with $y\geq0$. **Treating y as a function of x, define f(x) as the unique y such that $y\geq0$ and**

(1) $x^2+y^2=9$.

Hence, for $|x|\leq3$,

(2) $x^2+(f(x))^2=9$.

Let's call the function defined on the left hand side of equation (2) 'm' and the function on the right hand side 'r'. So $m(x)= x^2+(f(x))^2$ and $r(x)=9$. Notice that m(x) and r(x) are both functions of x and that (2) states that they are equal on the interval $0\leq x\leq3$. From this statement of function equality, we can conclude that r and m have the same rate of change on this interval. So when we take the derivative of both sides we maintain equality on this interval. That is:

(3) $m'(x)=2x+2f(x)f'(x)=0=r'(x)$ for $0\leq x\leq3$

In the above case we get that:

(4) $f'(x)=-x/f(x)$

The conceptual steps involved in legitimately making the inference of taking the derivative of both sides (transition from (2) to (3) above) appear below in Figure 2:

> **1.** Defining *f* by using (1).
> **2.** Viewing both sides of the equation as functions (of x).
> **3.** Recognizing that the functions defined by the left hand side and the right hand side are equal on the relevant interval.
> **4.** Recognizing that, since the functions are equal on an interval, the respective derivatives of the functions are also equal on that interval.

Figure 2. Conceptual steps involved in implicit differentiation, solving Fig 1 part a.

It bears mentioning that both Thurston (1972) and Staden (1989) noted that the legitimacy of the derivative of both sides procedure is rooted in function equality, although, Staden (1989) referred to these statements of function equality as "identity statements." However, as discussed in the introduction, much of the current education research on related rates and implicit differentiation problems overlooks this issue.

We pause briefly to discuss notation usage. At this stage in the manuscript we have solved problem (a) twice. The introduction used Leibniz notation (d/dx) in its solution to (a). This is consistent with popular calculus textbooks, where "taking the derivative of both sides" is often equated with "taking d/dx of both sides" (e.g., Rogawski, 2011; Stewart, 2006; Weir, et al., 2011). However, as we illustrated above, the reason taking the derivative of both sides is a legitimate procedure stems from the equation under consideration expressing function equality on some interval. Viewing the equation this way requires viewing the equation as (implicitly) defining a function; in the above, f(x) is implicitly defined in terms of its relationship to x^2 and 9. Hence the label "implicit differentiation". The role of function equality - in fact, the role of functions - is obscured by the procedural emphasis and use of Leibniz notation. With Leibniz notation, there are no functions explicitly under consideration. With the standard function notation used in the conceptual analysis, it is more apparent which functions are being differentiated and that f(x) is being implicitly defined. Further, some research suggests that students need to see equations written in standard function notation before differentiating (Engelke, 2008).

When Does the Equation Serve as a Function Definition?

The two problems in Figure 1 look very similar to each other; they have similar solution procedures that involve taking the derivative of both sides of the same equation,

(*) $$x^2+y^2=9,$$

and then applying derivative rules accordingly. However, the underlying reasoning that justifies the validity of performing a derivative operator on an equation differs between the two. Specifically, it is more involved to conceptualize (*) as a statement of function equality in **(a)** than it is in **(b).** In **(a),** the equation (*) not only asserts equality of functions, *but also "implicitly" defines a function* (the function f, discussed **in bold** above). In order to make sense of (*) as asserting a statement of function equality in terms of functions of x, one must conceptualize (*) as defining f and viewing y as equal to f(x).

The relevant functions in **(b)** are functions of time (not of x), since the task is to find a rate of change of distance with respect to time (t). Unlike with **(a),** Conceptual Step 1 is unnecessary; one does not need to conceptualize (*) as defining a function in order to view it as asserting a statement of function equality.

Given our previous discussion of the limitations of Leibniz notation, we use standard function notation in the remainder of this discussion. The letters *x* and *y* are shorthand for functions of time, x(t) and y(t), respectively. So for all t:

(**) $(x(t))^2+(y(t))^2=9.$

Similar to our earlier discussion, if we give the functions on the left and right side of the equation labels, say $m(t)=(x(t))^2+(y(t)^2)$ and r(t)=9, respectively, then (**) simply asserts that the functions *m* and *r* are equal for all values of *t.* Using similar reasoning to that of the previous problem this statement of function equality implies that m'(t)=r'(t). So:

(***) $2x(t)x'(t)+2y(t)y'(t)=0$

which, since we know y'(t)= -0.1m/s, yields:

(****) $x'(t)=(0.1y(t)/x(t))$

Notice that unlike (a), (b) entails only conceptual steps 2-4 from Figure 2, as there was no function of *t* implicitly defined by the equation. In other words, in problem (a), (*) simultaneously serves the purposes of both asserting a statement of function equality *and* implicitly defining a function. In problem (b), (*) only serves the purpose of asserting a statement of function equality. In this sense, only (a) truly involves implicit differentiation. In both situations, students must conceive of an equation as asserting function equality; however, to conceive of (*) as a statement of function equality involves first conceiving (*) as defining a function. Hence, it seems reasonable that problems like (a) might be more conceptually difficult for students than problems like (b). Defining the function *f*, as in (2), although perhaps trivial to mathematicians, could be a conceptual obstacle for students. Notice that (2) takes the form of "f(x) is the unique *y* such that the proposition P(x,y) is true." Being able to conceive of a function definition that involves outputs according to whether or not a proposition is true requires a process conception of function, which many students lack (Breidenbach, Dubinsky, Hawks, & Nichols, 1992). Unfortunately, it is common for educators to treat the two types of problems in Figure 1 synonymously as applications of taking the derivative of both sides of an equation, without attending to the meaning of the equation or the legitimacy of such an operation (discussed later).

To summarize, in order for one to understand the legitimacy of differentiating an equation, one must have a robust understanding of the equation itself. This robust understanding should involve viewing the equation as asserting a statement of function equality (Conceptual Step 3), which requires viewing each side of the equation as defining a function (Conceptual Step 2). As argued above, in the problem in Figure 1b, Conceptual Step 2 is *easier* than in 1a, as Conceptual Step 1 is not involved. Our reason for carefully contrasting the two types of problems in Figure 1 is to emphasize that, while these problems have similar procedural solutions, when attending to the legitimacy of differentiating the equation, they are not the same.

We are *not* claiming that the only conceptual work involved in understanding implicit differentiation and related rates problems is in understanding the legitimacy of differentiating an equation. This is just the conceptual aspect that we choose to focus on in this paper, as it has largely been ignored so far in the literature. Now that we have provided the reader with a conceptual basis for understanding implicit differentiation/related rates problems, including the conceptual steps required to make sense of the legitimacy of these procedures, we shift to discussing the literature.

Literature

We searched the literature thoroughly by reviewing every available RUME paper that included the words "implicit differentiation" or "related rates", as well as every paper in the online archives of The Journal of Mathematical Behavior, Mathematics Teacher, Journal for Research in Mathematics Education, Mathematics Education Research Journal and all other National Council of Teachers of Mathematics publications. A Google Scholar search was also performed, but the methodology for that search was not recorded. Despite this expansive search, only two articles (Thurston, 1972; Staden, 1989) address the legitimacy of differentiating both sides of an equation. In both articles, the topic is only mentioned in passing, and there is no discussion of student understanding. Staden (1989) specifically argues that students are "mistaught" by being told that they can differentiate each side of an equation (when, as discussed above, this does not work for *any* true equation), and suggests that students might have resulting misunderstanding.

The remainder of the literature tends to treat differentiating an equation as only a procedural aspect of implicit differentiation or related rates problems. In fact, many authors appear to treat "implicit differentiation" to mean something like "using Leibniz notation while differentiating an equation", not distinguishing true implicit differentiation (like **(a)**) from differentiation in related rates problems that's not truly implicit (like **(b)**) (Jones, 2017; Martin, 2000; Engelke, 2007; Garcia & Engelke, 2013). This is unsurprising when we consider that, when viewed procedurally, differentiating with respect to x and with respect to t is almost identical. Hare and Philippy (2004), for example, write a lesson plan outline that includes the assertion "Implicit differentiation must be used whenever the differentiation variable differs from the variable in the algebraic expression (p.9)" and stresses use of the chain rule. If one is not attending to the rationale for differentiating, then attending to the "differentiation variable" and when to use the chain rule is similar in problems like **(a)** as in problems like **(b)** (in **(a)**, the "differentiation variable" is x, and in **(b)** it is t).

Martin (2000) provides a "problem-solving framework" characterizing the steps in solving related rates problems similar to **(b).** She not only conflates "implicit differentiation" with "taking d/dx of both sides," but also overtly labels differentiating each side of an equation as "procedural." Engelke (2007) utilizes Martin's framework to develop a "mental model"; this mental model further de-emphasizes the conceptual aspect of differentiating equations by consolidating Martin's "implicitly differentiate" step with another step to create what she calls a "phase". When we consider how Martin created her framework, it is unsurprising that she does not address the legitimacy of "implicit differentiation"; she created the framework by observing written solution *procedures* to related rates problems. Since conceptualizing a justification for differentiating an equation is not a procedure, it makes sense that it would remain unaddressed. This is not to suggest that Martin's model is not useful, only that it leaves this particular matter unaddressed.

Student Confusion

We have established, that both common textbooks and the majority of mathematics education literature ignore the conceptual basis for implicit differentiation. However, we realize that some might view this as unproblematic. In this section we take a brief look at some data to establish that the lack of student understanding of the conceptual basis for implicit differentiation. A search of popular online student help forums, Khan Academy and Stack Exchange, suggests that students are unclear of the validity and meaning of applying the differential operator to each side of the equation (Anonymous, n.d.; Frank-vel, 2015; Jon, 2013; Klik, 2013; Mathematicsstudent1122, 2016; Ryan, 2016; Wchargin, 2013). Further, the work of one of the authors of this manuscript suggests that a strong understanding of function equality may be absent in a number of calculus students (Mirin, 2017a; 2017b).

In order to learn more about students' understandings of the conceptual steps involved in implicit differentiation (Figure 2), a student, John, was interviewed by the first author of this manuscript. He was enrolled in Calculus II at Anonymous State University (ASU) and had taken Calculus I, which includes a unit on implicit differentiation, the semester prior. The interview was a semi-structured clinical interview and lasted an hour (Hunting, 1997). Throughout the interview, John was asked to think about ideas regarding implicit differentiation and function equality that he had perhaps not reflected on before. John might have never considered these matters, and might therefore have made on-the-spot explanations.

The interview centered around four prompts:

Prompt 1. What is your meaning for implicit differentiation? How do you interpret the word "implicit" in this situation?

Prompt 2. Find dy/dx for $x^2+y^2=1$ when $y>0$

Prompt 3. A 10-foot ladder leans against a wall; the ladder's bottom slides away from the wall at a rate of 1.3 ft/sec after a mischievous monkey kicks it. Suppose h(t)= the height (in feet) of the top of the ladder at t seconds, and g(t)=the distance (in feet) the bottom ladder is from the wall at t seconds. Then $(h(t))^2-100=-(g(t))^2$. How fast is the ladder sliding down the wall?

Prompt 4. True or false: Suppose f(x)=g(x) for all values of x. Then f'(x)=g'(x).

Figure 3: The prompts that formed the basis for the clinical interview.

The interview lasted an hour. Due to space constraints only the most pertinent highlights are reported here.

John expressed that he did not remember exactly what the procedure of implicit differentiation was, but that it was something that must be done when there is no function (i.e., due to failure of the vertical line test). He did not have an idea of what the *implicit* referred to in implicit differentiation. John did not have an idea of how to approach Prompt 2, so the interviewer reminded him of a procedure that was done in his Calculus I class: replacing *y* with f(x) before differentiating the equation and that $x^2+y^2=1$, $y>0$ defines the top half of a circle, and asked him to elaborate on what $x^2+(f(x))^2= 1$ means. He explained that 1 is the radius, and having f(x) [in place of *y*] "makes the computation easier". He was then asked him explicitly what it means for the right hand side of $x^2+(f(x))^2=1$ to equal the left hand side, and he responded "It's a circle. I just see a circle." When prompted to explain what the circle has to do with the equation, he graphed two parabolas - a sideways parabola (representing y^2) and an upright parabola (representing x^2) and asked "how is that a circle?". In this situation, it seems that John was not thinking of y (or f(x)) as a function of x; instead, he seemed to be thinking of "y^2" as denoting the parabola that he associates with "$x= y^2$". After reasoning with a graph was unhelpful to John, he began considering specific values of x and y, observing that "as they change together, in this equation here, they have to change together in such a way that it always equals 1."When asked about the legitimacy of taking d/dx of both sides, he drew an analogy to algebra: "If I have x=1, I multiply by 2 and get 2x=2, it would be the same thing." He related the procedure of taking d/dx to inferring equal rates of change: "if you take the rate of change of this [left hand side], it is the rate of change of this [right hand side]. They're equal to each other, so the change in one is gonna be the change in the other." Since John believed the inference of equal rate of change came from *something* being equal, to get at what that *something* was, the interviewer asked him what happens if he differentiates each side of x=1. He noticed that it results in 0=1, which he said did not make sense.

The interview then shifted to Prompt 3. John was reminded that he could take the derivative of both sides of the equation, and he did so with some minor errors. He explained that the distance the ladder is from the wall, g(t), and the distance the ladder is from the floor, h(t), "change together". Even when pushed, he did not say why taking the derivative of both sides is a valid procedure. Instead, John continued to express an understanding of the two distances as changing together as time changes, and failed to mention each side of the equation as representing a function:

> "We take the derivative of both sides because...you need to have the two rates change together, in order for this scenario to work. Because if they don't with respect to each other, then uh...it just doesn't hold true. So we do it on both sides in order to have the scenario change together and everything stay true to itself...maybe."

Since John was not using the language of functions on his own, the interviewer decided to move to Prompt 4 in order to see if he could relate taking the derivative of each side of an equation to an inference from function equality. John almost immediately provided what he viewed as a counterexample to the assertion that if two functions are equal, then their derivatives are also equal. By misapplying the quotient rule, he argued that f(x)=x and g(x)=2x/2 are equal for all values of *x* but have different derivatives. He explained that, if he were to simplify g(x), he would end up with the same derivative as that of *f*, but that simplification before finding derivatives is not permitted. This highlights that John had a fundamental misunderstanding of how the derivatives of two equal functions relate, a key aspect in understanding the legitimacy of applying the derivative operator. We believe this misunderstanding contributed to his struggles with making sense of why the implicit differentiation procedure is legitimate.

Discussion

We have performed a conceptual analysis of the implicit differentiation procedure. We have established the conditions under which taking the derivative of both sides of an equation is legitimate, why it is a legitimate procedure under these conditions and when a function is implicitly defined. In the conceptual analysis this process is broken down into 4 conceptual steps, which may form the basis of instruction aimed at better student understanding of implicit differentiation. Only 3 of these steps are needed to make sense of related rates problems. We showed that the way of understanding described in the conceptual analysis is largely absent from the mathematics education literature, which in turn bolsters the need for this analysis**.** This points to the fact that the understanding developed in the conceptual analysis may be non-trivial to develop in students. Finally, the study reports a brief excerpt from a successful calculus student John, which establishes that the understanding developed in the conceptual analysis is not present in some students and is non-trivial to develop. In future research we aim to explore this issue in more detail by conducting a multi-student teaching experiment aimed at developing rich student understanding of implicit differentiation.

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