“Derivative makes more sense with *differentials*”: How primary historical sources informed a university mathematics instructor’s teaching of derivative[1]

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Abstract

In this brief research report, we address the recent calls to improve undergraduate mathematics instruction through our investigation of an instructor’s teaching of derivative in a Calculus course. Considering his efforts to modify the presentation of derivative in the textbook as attempts to improve his teaching as a result of his engagement with primary historical sources, we analyze his teaching to identify the changes in his practice by using Speer, Smith, and Horvath’s (2010) framework for “teaching practice.” With our analysis of instructor interviews and video-recordings of classroom sessions, we observe that Leonhard Euler’s use of differentials in defining derivative had responded to his pedagogical concerns, and had convincing power as a method, which, in turn, led him to make significant changes in how he selects and sequences content for his teaching.

Keywords: Primary Historical Sources, Derivative, Calculus, Teaching Practice

Introduction

When called upon improving their instruction, instructors of undergraduate mathematics are suggested to “present key ideas and concepts from a variety of perspectives, employ a broad range of examples and applications to motivate and illustrate the material, promote awareness of connections to other subjects, and introduce contemporary topics and applications” (Saxe & Braddy, 2015, p. 1). Given the importance of classroom mathematics teaching for student learning (Hiebert & Grouws, 2007) and the many challenges that an instructor has to deal with regarding, how can an instructor of mathematics ensure the completion of tasks that are expected from her? How can she manage to create learning environments that are meaningful to her students for every class that she is supposed to teach?

In this regard, a more important question to ask is about the support that instructors receive, rather than expecting them to meet the needs of students, departments, and institutions on their own. Instead, given the complexity and difficulty of teaching mathematics in itself along with all the logistics that an instructor has to deal with to create a learning environment, regardless of instructor’s philosophical or theoretical orientation towards pedagogy of mathematics, instructors should get, to list a few, logistical, curricular, and motivational support. In this preliminary report, we are arguing for the ways that primary historical source can encourage, inform, or guide teaching practice for the teaching of undergraduate mathematics.

Problem Domain and Purpose

The primary motivation of this article is Speer, Smith, and Horvath’s (2010) call for more empirical research on collegiate mathematics instructors’ teaching practices. To better situate their discussion of instructors’ practices, Speer et al. distinguish instructional activity and

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teaching practice. "*Instructional activities* are the organized and regularly practiced routines for bringing together students and instructional materials" (emphasis added, p. 101). Lecture, small group work, or whole-class discussion are some of the commonly used instructional activities at the undergraduate mathematics education. Accordingly, teaching practice refers to "teachers' *thinking*, *judgments*, and *decision-making* as they prepare for and teach their class sessions, each involving one or more instructional activities" (p. 101). Furthermore, we are interested in a situation where history of mathematics was used in a specific way. In Jankvist's (2009) terminology, this report is concerned with the "hows" but not the "whys" of using history of mathematics.

The purpose of this article is to share some preliminary findings of our research where we investigated the ways that primary historical sources can inform, guide, or inspire the teaching practices of a university mathematics instructor in his teaching of derivative. Through sharing the story of an instructor and making the changes in his teaching practices explicit as a result of his engagement with primary historical sources, we are aiming to contribute to "our understanding of collegiate mathematics teaching and of the resources that collegiate teachers, especially beginners, might access to learn about the work of teaching" (Speer et al., p. 99). The research question that our investigation was based on is the following:

In what ways, do primary historical sources, inform, guide, or support a university mathematics instructor's teaching practices for the teaching of derivative in the first course of the Calculus sequence?

Theoretical Framework

Our use of a theoretical framework in this report is to explore the teaching practices of an instructor and how they change as a result of his engagement with primary historical sources, rather than to discuss the effectiveness of such an engagement for student learning. In particular, we use Speer et al.'s (2010) framework on teaching practices to describe the practices of an instructor in his attempts to teach derivative as a result of his engagement with primary historical sources. Due to the space considerations, we only share results for only one component of the framework. There are seven dimensions of teaching practice that are identified by Speer et al. The one that we are interested in this proposal is italicized (a) Allocating time within lessons, (b) *Selecting and sequencing content (e.g., examples) within lessons,* (c) Motivating specific content, (d) Posing questions, using wait time, and reacting to student responses, (e) Representing mathematical concepts and relationships, (f) Evaluating and preparing for the next lesson, and (g) Designing assessment problems and evaluating student work. In this brief report, we are able to analyze our data for one aspect.

Selecting and sequencing content. This component of the framework refers to the content to be taught for a course, the order of topics through the semester, and examples/exercises to be shared with the students are some of the aspects of how an instructor selects and sequences content. As Speer et al. (2010) noted, although instructors mostly rely on textbooks for this aspect of their teaching practice, there are times that instructors may decide to consider, for instance, omitting some parts of a chapter in the textbook, provide students with examples from a different source, or create her own set of exercises for her students.

Methodology

Our research is a result of our interest in an instructor's attempts to modify his teaching of derivative based on his engagement with primary historical sources. Our goal is to provide in-depth description of the experiences and views of the instructor to better demonstrate his interactions with the primary historical sources, and how such interactions led him to reconsider his teaching of derivative. Therefore, our inquiry in this research is qualitative in nature and descriptive by purpose. In Stake's (1998) terms, we identify our research as an intrinsic case study: a result of our interest in the story of an instructor, rather than trying to understand a phenomenon.

The participant of our study is a male mathematics instructor, from now on we call T, who was at his first semester in teaching at a tenure-track faculty position at a university located at Central region in the United States of America. Our data is on his teaching of derivative in the first course of Calculus sequence. Although this was his first semester in teaching Calculus as a faculty member, he had three semesters of experience in teaching Calculus as a doctoral student. His first interaction with primary historical sources is through one of the instructional materials known as Primary Source Projects (PSPs). To describe briefly, a PSP is a curricular material aiming to guide students' reading and study of selected excerpts from primary historical sources. (see Barnett (2012) and Barnett, Lodder, and Pengelley (2014) for detailed information on PSPs.)

The PSP that T used for his teaching is *The derivatives of the sine and cosine functions* (Klyve, 2017), which is designed for two class sessions of teaching. This PSP includes excerpts from Leonhard Euler's *Foundations of Differential Calculus*. Through some excerpts from Euler (1755) and tasks related to these excerpts, Klyve, first, aimed to familiarize students with how Euler used differentials. Consequently, the goal was to share an alternative approach to the limit definition of derivative, where Klyve, eventually, provided how Euler used *differentials* to calculate the derivative of the sine function. Our decision to conduct research on T's teaching on derivative began with his decision on extending the idea of using differentials for the entire derivative chapter of the course. The following quote is used in the PSP as an excerpt from Euler's original text to demonstrate how differential was calculated, and derivative was defined using differentials.

> From this fact there arises a question; namely, if the quantity x is increased or decreased, by how much is the function changed, whether it increases or decreases? For the more simple cases, this question is easily answered. If the quantity x is increased by the quantity ω, its square x^2 receives an increase of $2x\omega + \omega^2$.
> Hence, the increase in x is to the increase of x^2 as ω is to $2x\omega + \omega^2$, that is, as 1 is to $2x + \omega$. In a similar way, we consider the ratio of the increase of x to the increase or decrease that any function of x receives.
> Indeed, the investigation of this kind of ratio of increments is not only very important, but it is, in fact, the foundation of the whole of analysis of the infinite.
> In order that this may become even clearer, let us take up again the example of the square x^2 with its increment of $2x\omega + \omega^2$, which it receives when x itself is increased by ω. We have seen that the ratio here is $2x + \omega$ to 1. From this it should be perfectly clear that the smaller the increment is taken to be, the closer this ratio comes to the ratio of $2x$ to 1. (Klyve, 2017, p. 2)

We collected data through pre-semester and post-semester surveys, interviews, and video recordings of selected classroom sessions. If the instructor believed that he would spend time on differentials, we decided to include that session for video-recording. Each class session was 75 minutes, and ten out of 30 sessions were video-recorded. We conducted two interviews with the instructor: One at the beginning of the semester, where our goal was, basically, to develop an understanding of him regarding his perspective on mathematics, pedagogy, Calculus and its teaching, and his experience with history of mathematics. Second interview was conducted at the end of the semester. We asked him to reflect on his experience by asking some specific questions on his teaching practice. In our analysis, we mainly rely on interviews and video recordings to describe and understand his teaching practice.

We analyzed data, primarily video-recordings in this case, to observe the changes in the teaching practice based on instructor's description of his regular and planned teaching of derivative. In this analysis, we also paid attention to discovering the potential role of his engagement with the PSP and Euler (1755) on the changes in his teaching practice. We discussed our observations and interpretations with the instructor for the validation of findings.

Findings

Although T was about the begin teaching as a faculty member for the first time, he had observed extensively in his prior experience in teaching Calculus that students used to struggle in understanding the concept of derivative. As he stated in the pre-interview, one of the most notable challenges that students experienced was the limit definition of derivative, which was also a challenge for him when he was a mathematics major in his undergraduate program.

T's initial decision to use a PSP for his teaching relied on his interest in the history of mathematics. However, he had never used any primary historical document for his teaching prior to his experience with PSPs. When he decided to make use of the opportunity of using PSPs for his teaching, T's goal was primarily to supplement his teaching with the textbook, which was supposed to take two class sessions. However, as we share in our further analysis, T ended up making fundamental changes in his teaching after his engagement with the PSP.

When asked about his first reaction after his first reading of the PSP, T stated that he was very surprised with the emphasis given on differential in defining derivative concept since differential was mentioned in the last section of the nine sections in the derivative chapter of the Calculus textbook. In his words during the pre-interview, "but when you look at the textbook, there are nine sections in derivative chapter and the differential section is the last section." He continued as follows to describe his reaction to the importance given on the differential in the PSP:

> When I look at my previous experiences, students cannot really learn the definition of the derivative, limit definition of the derivative and they just memorize the formula. […] They do not really learn what is going on, why that formula works, what that *dx* means in the formula. But […] when I define the derivative using differential and when I first explain the idea of using differential to them, and then using that idea to computing and defining the derivative, I believe and I expect […] they will really understand what is going on in the definition of the derivative and what the derivative is.

In this regard, it is important to note that differential as a mathematical idea central to the definition of derivative provided the instructor with a vocabulary so that he believed that he had the tools to communicate commonly used symbols in derivative, *dx* and *dy*, with students in meaningful ways. For instance, using differentials in defining the derivative allowed T to provide a justification for why Leibniz's notation in chain rule makes sense, and why it works, first of all, for himself as an instructor of mathematics. To us, Euler's approach, using differentials to define derivative, allowed T to produce narratives on derivative that, first, convinced him as a learner of mathematics. Therefore, he proceeded with modifying his teaching practice expecting that Euler's approach would also support a meaningful conceptualization of derivative.

Next, we share the significant changes in T's teaching practices as a result of his engagement with the PSP and Euler's approach for defining derivative using Speer et al.'s (2010) framework. Due to the space considerations, we report our findings on *selecting and sequencing content* aspect of that framework.

Selecting and Sequencing Content

In the pre-interview, when asked about how he used curricular materials, in particular textbook, informed his teaching, T told that textbook would be the main guide for his teaching in planning and delivering his lectures. The textbook used to dictate, as he expressed, almost all of his teaching practices, including how he defined the concepts, the examples he used to explain mathematical ideas, and the exercises that he asked students to work on in his prior teaching experiences, and would dictate if he did not meet Euler's approach.

Following his interaction with the PSP (Klyve, 2017), and Euler (1755), T did not only replace the limit definition of derivative with Euler's approach using differentials, but also, he redesigned each section in the derivative chapter of the textbook. For instance, he used differentials while introducing the differentiation techniques for the derivatives of constant and identity functions. As another example, in the product and quotient rule section, he said in the class, "we will go back to 1700s and visit Euler in his office and ask him how we can take the derivative of product of two functions. Let's see what he is doing" and showed what those rules are and why they work while using the differential approach. He used the Leibniz notation as the primary representation for derivative in his teaching. Associating it with the phrase "crucial word," he used to call "the ratio of $\frac{dy}{dx}$," as the "magical ratio."

Therefore, we conclude that T's engagement with the PSP (Klyve, 2017) and Euler (1755) provided him a different perspective on conceptualizing, defining, and introducing derivative, which ended up with significant changes in how the content was presented to students.

Discussion Questions for Further Analysis

For our work in this report, we found Speer et al.'s (2010) framework as an effective tool to explore the teaching practices of a mathematics instructor and investigate the changes in his practice as a result of his engagement with a PSP (Klyve, 2017) and Euler (1755). Clearly, T's interest in the history of mathematics was influential on his interest in using Klyve's PSP. However, based on what our data suggests, we argue that it was Euler's approach that triggered the changes in the teaching practice. Although the effectiveness of these changes in teaching practice on student learning is a question of interest, we believe that finding a motivation for instructional change is noteworthy.

In this regard, we highlighted the role of primary historical sources in this proposal, but we also believe that further research needs to consider instructor characteristics as an aspect of investigation to deepen our understanding on the dynamics of change in teaching practice.

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