The Transfer and Application of Definitions From Euclidean to Taxicab Geometry: Circle

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Research shows that definitions in mathematics are often not used correctly by students in mathematical proofs and problem-solving situations. By observing properties and making conjectures in non-Euclidean geometry, students can better develop their understanding of these concepts. In particular, Taxicab geometry is suggested to be introduced before other non-Euclidean geometries since it is a considerably simpler space. To further investigate this, APOS Theory is used as the framework in this analysis of responses to a real-life situation from students enrolled in a College Geometry course at a university. Through the perspective of APOS Theory, this report provides two representative illustrations of the conceptual understandings found among these students in relation to the definition of circle. By adapting and applying their knowledge of definitions from Euclidean geometry to Taxicab geometry, these students provided insight as to how Taxicab geometry concepts are assimilated into their existing understanding of concepts in geometry.

Key Words: Definitions, Geometry, Taxicab, Circle, APOS Theory

Introduction

Edwards and Ward (2008) found mathematics majors exist that do not understand the role of definitions in a mathematically acceptable way but have been deemed successful students in advanced mathematical courses. The authors explain that this should be addressed in undergraduate mathematics, and research is needed to determine pedagogical strategies that help facilitate student's understanding of the concept of definition. Emphasizing the importance of definitions in geometry, Güner and Gülten (2016) explain that geometry has three dimensions: definitions, images that represent these definitions, and their properties. In such context, since the properties of geometric figures are derived from definitions within an axiomatic system, it is important to note that a figure is "controlled by its definition," (Fischbein, 1993, p. 141).

In college geometry courses, Euclidean geometry and its axiomatic system is thoroughly studied, but other axiomatic systems receive little consideration (Byrkit, 1971; Hollebrands, Conner, & Smith, 2010). This is despite the fact that research shows by exploring concepts in non-Euclidean geometry, students can better understand Euclidean geometry (Dreiling, 2012; Hollebrands, Conner, & Smith, 2010; Jenkins, 1968). One example of a non-Euclidean geometry in which students can explore concepts is Taxicab geometry. This is the geometry that is the result of measuring distance as defined by the L_1 norm. Siegel, Borasi, and Fonzi (1998) encourage the introduction to Taxicab geometry before other non-Euclidean geometries since the simpler space makes it easier for students to reason and thus abstract concepts. Consistent with this claim, Dreiling (2012) found that "through the exploration of these 'constructions' in Taxicab geometry… [students] gained a deeper understanding of constructions in Euclidean Geometry," (p. 478). For this report, we present results and discussion on the following research question: In what ways do students assimilate the definition of a *circle* in Taxicab geometry into their existing understanding of this concept?

Theoretical Framework

APOS Theory is a constructivist theory based on Jean Piaget's theory of reflective abstraction, or the process of constructing mental notions of mathematical knowledge and objects by an individual during cognitive development, (Dubinksy, 2002). In APOS Theory, there are four different stages of cognitive development: Action, Process, Object, and Schema (Arnon et al., 2014). In addition, there are mechanisms to move between these stages of cognitive development, such as interiorization and encapsulation. An Action in APOS Theory is being exhibited when a student is able to transform objects by external stimuli or perform steps to complete a transformation. As a student reflects on Actions, they are able to *interiorize* them, so they can imagine performing these Actions without actually doing so. In this case, we refer to interiorized actions as a Processes. A student can then *coordinate* processes with others within a schema in order to form connections between concepts. Once a student is able to think of a Process as a totality to which Actions or other Processes could be applied, we say that an Object is constructed through the *encapsulation* of the Process. Finally, the entire collection of Actions, Processes, Objects, and other Schemas that are connected to the original concept that form a coherent understanding is called a Schema (Dubinsky, 2002). We provide examples of evidence of the stages of cognitive development in APOS Theory for the concept of **Radius** within the context of this paper. When given two points and asked to find the center of a circle such that these two points are on the circle, if an individual does so by counting blocks or guessing and checking at radii lengths until they find an appropriate one, they are exhibiting an action conception of **Radius**. In the same scenario, finding the total distance between both points and dividing by two provides a possible radius for a circle whose center is equidistant from the two points. In this case, the individual is exhibiting a process conception of **Radius.** We note there are an infinite number of circles that can be constructed such that two given points are on the circle. If a student is aware of this and explains this implies there is more than one radius measure for which such a circle can be constructed, he or he is exhibiting an object conception of **Radius** since the student is performing an action of comparison on his or her **Radius** object.

A genetic decomposition is defined as a "description of how the concept may be constructed in an individual's mind," (Arnon et al., 2014, p. 17). For this study, a genetic decomposition was developed to identify development pathways students may follow to adapt their working understanding of the definition of a circle to incorporate concepts in Taxicab geometry. In other words, this report focuses on how students assimilate the concept of **Circle** in Taxicab geometry into their existing *circle schema.* The subconcepts of **Circle** as defined in this report are **Distance, Radius, Center,** and **Locus of points.** In order to construct a **Circle** process, a student must have a process conception of at least two of these subconcepts. Figure 1 shows a possible way a student can construct a **Circle** process by the coordination of his or her **Distance, Radius, Center,** and **Locus of points** processes.

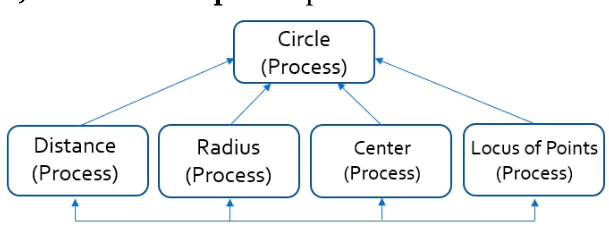

Figure 1. A construction of the Circle process.

A possible pathway a student may take in order to assimilate the concept of a circle in Taxicab geometry into their *circle schema* is shown in Figure 2. We show this assimilation using the subconcept of **Distance** but note that each of the subconcepts mentioned prior is expected to be assimilated into the *circle schema* in a similar manner.

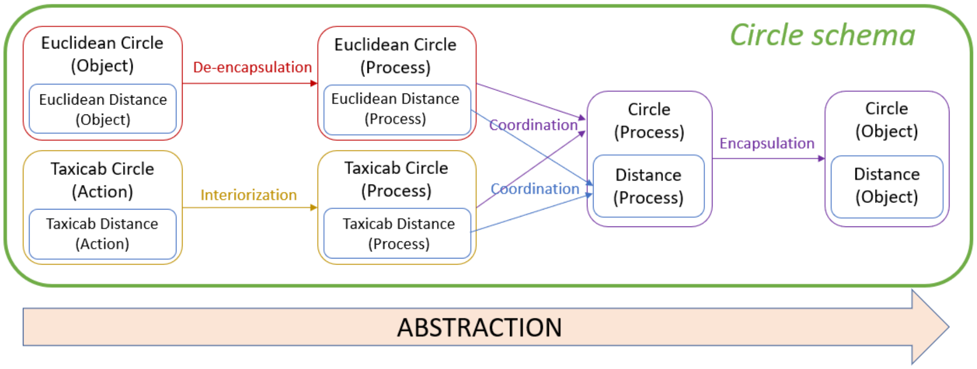

Figure 2. A possible way a student may assimilate Taxicab distance into his or her circle schema.

Methodology

This research study was conducted at a university in a College Geometry course during a Fall semester, which has an introduction to proof course as a prerequisite. Since it is a cross listed course, there were both undergraduate and graduate students enrolled in the course. The textbook used in the course was College Geometry Using the Geometer's Sketchpad (Barbara E. Reynolds & William E. Fenton, 2011), written on the basis of APOS Theory. This study consisted of sessions of instruction on Taxicab geometry by one of the authors of this report, followed up with interviews conducted by the other author. The material of the course covered concepts and theorems in Euclidean geometry often seen in a College Geometry course and included Taxicab geometry for four 75-minute class sessions at the end of the semester. Written work from the semester and videos from the in-class group work and discussion during the Taxicab geometry sessions were collected and used as data in the study. After the semester but before final exams, semi-structured interviews were conducted with participants from the course. These interviews were conducted with 15 of the 18 students enrolled in the course who voluntarily signed up to participate in the interviews. All 18 students consented for their in-class group work and discussion to be recorded, as well as written work and exams throughout the course to be collected. Results from the analysis of student responses to a question on the final exam pertaining to concepts in Taxicab geometry are presented below within the context of APOS Theory. We focus our attention in this paper to responses from two of the 18 students enrolled in this course who were both secondary mathematics teachers and graduate students. We note prior to presenting results that students learned a continuous model of the Taxicab metric (or the L_1 norm). That is, distance between two points is measured continuously, not discretely. The problem on the final exam was stated as follows:

> Assume [a university's] campus and surrounding streets are designed explicitly in a grid pattern, i.e.- distance is measured by Taxi-distance. You are looking for an apartment near campus, but you want to make sure that from your apartment, the walking distance to [Building 1] (located at (-2, -2)) is the same as the walking distance to the [Building 2] (located at (4, 3)), since you have classes in both locations.
>
> a. Draw a graphical representation of where your apartment could be located, given that it needs to be equidistant from [Building 1] and [Building 2].
> b. What mathematical term would describe what you have drawn in your sketch?

The expectation for this problem (and an ideal solution) would be for students to recognize that there are an infinite number of places they could have an apartment so that its location is equidistant from the two buildings, with a midpoint having the shortest distance. Further, students should identify that the set of points equidistant (in Taxicab geometry) from both buildings is the equivalent of the Euclidean perpendicular bisector of the segment connecting the two buildings. The problem was open-ended without explicitly asking for students to identify a specific location for their apartment, but rather asked them to draw a graphical representation of the problem. For this reason, responses were expected to vary with regard to what mathematical term students associated with their drawing.

Results

As representative illustrations, we provide the APOS Theory based analysis of Kym's and Hannah's solutions to the exam problem as they correspond to this preliminary genetic decomposition.

Provided in Figure 3 is Kym's solution to this problem on the Final Exam. Kym was a graduate student and secondary mathematics teacher enrolled in the course who also participated in the interviews prior to this exam. Note Kym seemed to be operating with a discrete model of Taxicab geometry, as evidenced by her note "let 1 unit = 1 block."

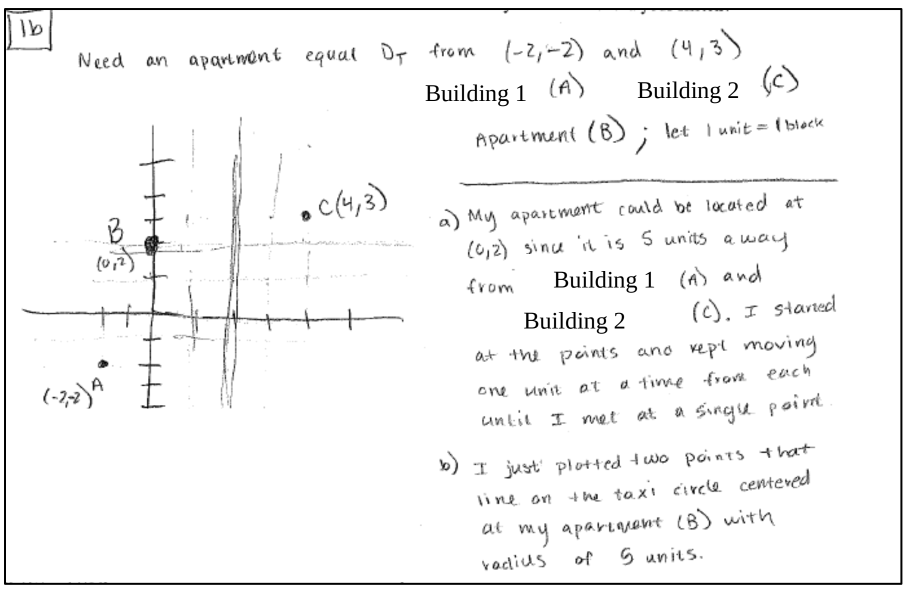

Figure 3. Part of Kym's solution to the given problem.

As we can see in the bottom right of Figure 3, Kym described her sketch as the Taxicab circle centered at her apartment with a radius of 5 units, mentioning prior that the two buildings would lie on this circle. Note that by saying she "plotted two points that [lie] on the taxi circle," she has in a way reversed the direction of the problem, since the problem was to find a point equidistant from these two points, not to plot two points equidistant from some fixed point. In any case, she demonstrated with this statement that she understood a Taxicab circle has this property of equidistance between the center and the points that lie on the circle. By saying in part (a) she "kept moving one unit at a time" to count out her distance between these points, it appeared that Kym was creating/constructing two radii of this Taxicab circle. Kym seemed to have at least a process conception of **Distance** and **Locus of points** since she could imagine a circle with the buildings lying on this circle. With the evidence provided in her solution, by stating in the past tense how she found this point/center (operating within the context of this problem) and counting blocks to define the radii of the circle, Kym was exhibiting an action conception of **Radius** and **Center.** Her solution point of $(0{,}2)$ was not actually the center of a circle with the buildings lying on the circle, since from her solution point to the buildings, one Taxicab distance is five and the other is six. However, we believe this inaccuracy was due to her operating with a discrete model of Taxicab distance.

Like Kym, the next solution presented was provided by a graduate student and secondary mathematics teacher who also participated in the interviews prior to the exam. The following was Hannah's solution where she provided an equation of this circle and exhibited an object conception of some of the subconcepts of **Circle** in a way not accounted for by the genetic decomposition.

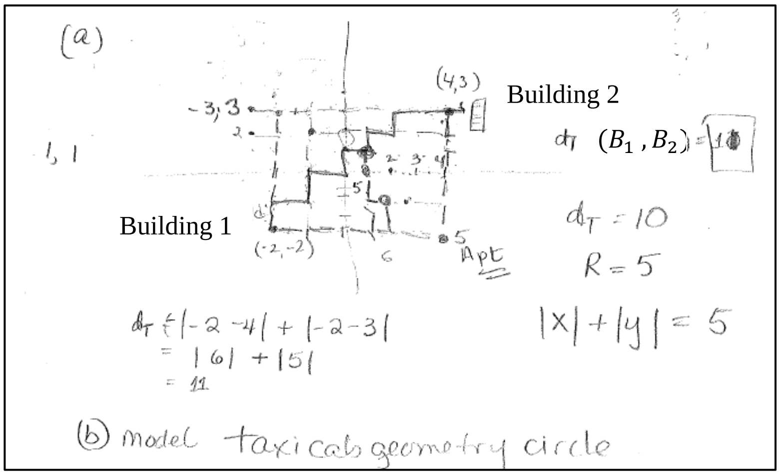

Figure 4. Hannah's solution to the given problem.

Seen in Figure 4 as her part (b), Hannah first described her sketch as a "model taxicab geometry circle." It appears as though Hannah first thought the distance between the buildings to be 10 units, as seen in the mid-right area of her work with what she wrote as "$d_T = 10$," although above this she corrected her initial calculation to be 11. This perhaps led to her labeling the radius to be 5 when she wrote "$R = 5$." A closer look at Hannah's drawing provides evidence she was attempting to find a center of a circle by constructing radii of length 5 or 6, with what looks like steps in her drawing. This is evidence that, like Kym, she most likely discretized the Taxicab metric which could be why she was counting radii of lengths 5 or 6. Hannah then tried to write the equation of the Taxicab circle she mentioned in part (b). By first finding a value for the radius and using this value to construct a possible circle, Hannah exhibited a process conception of **Radius.** Further, she attempted to plug this value into an equation of a Taxicab circle. Thus, Hannah had encapsulated her **Radius** Process into an Object since she was using it as an input into some function whose output was a Taxicab circle equation, performing an action on this Object.

Given the location she chose as her apartment of $(4, -2)$, indicated in the lower right of her graph as "Apt," the correct equation of this Taxicab circle would be $|x - 4| + |y + 2| = 5$. However, she wrote this equation as $|x| + |y| = 5$. It is possible with her equation she was attempting to indicate for the center of the circle and a point on the circle, "the change in x plus the change in y is equal to 5," but we do not have further evidence of this claim. Regardless, she was able to imagine that the solution would be the center of a circle, as indicated by her drawing and calculation of a possible radius for such a circle, which implies Hannah was exhibiting at least a Process conception of **Locus of points**. Although her equation of a circle is not indicative of a process conception of the algebraic representation of **Taxicab circle**, Hannah's geometric solution and approach to the problem provides evidence of at least a process conception of **Radius, Distance,** and **Locus of points** in Taxicab geometry and that she had coordinated some of these processes. Hannah also exhibited evidence of an object conception of **Distance** and **Radius**.

The mental structures necessary to write or derive the equation of a circle were not explicitly considered in the genetic decomposition. To write the equation of a circle, a student would need to identify an appropriate length for a radius and the center of a circle that is this distance from both buildings, specifying the metric used. It is possible that a student can write the equation for a circle but not understand or be able to explain how each part of this equation is a result of the definition of a circle and its subconcepts. In this case, he or she would most likely be memorizing a template for this equation. If the student has a process conception of several subconcepts, they may not have coordinated them with one another to make necessary connections to understand the equation's derivation. In this case, a student exhibits an object conception of all of the subconcepts of **Circle**, since he or she is using them as inputs into a mental function, but does not have a coherent understanding of the underlying structure of the *circle schema*. This understanding may be gained by the student *de-encapsulating* his or her object conceptions of each of **Distance, Radius, Center,** and **Locus of points** and coordinating them with one another to observe these relationships. An illustration of this is provided in Figure 5. In particular, the blue arrows indicate the de-encapsulation of all of these objects into processes. The red arrows in this figure indicate the possible coordination that could then occur among these processes.

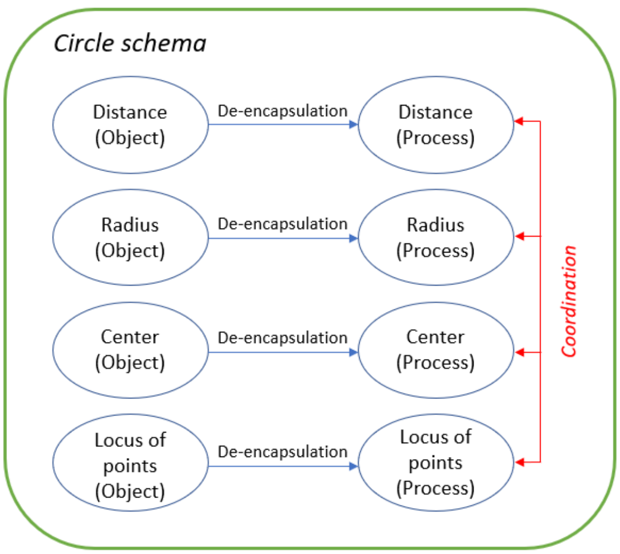

Figure 5. The de-encapsulation of objects to coordinate processes within the circle schema.

For this problem on the final exam, no students exhibited an object conception of all subconcepts of **Circle**. This may be a result of the manner in which the problem was stated since it did not necessarily require students to utilize an object conception. In the Discussion section, we present suggested questions that can help probe for this, as well as guide students in the construction of mental structures that are necessary to encapsulate these processes.

Discussion and Concluding Remarks

Fischbein (1993) explains that in geometrical reasoning, a major obstacle is the tendency to “neglect the definition under the pressure of figural constraints,” (p. 155). By designing a problem where a student is essentially told a definition and has to derive the associated mathematic term, we hoped to overcome this obstacle in that it would minimize any misconceptions a student may have associated with a concept. Supporting this notion, although Taxicab circles look different than Euclidean circles, Kym and Hannah were able to use their geometrical reasoning skills to arrive at a solution to the given problem. They did so by applying their knowledge of definitions to correctly identify a mathematical term that satisfies the conditions of the problem. This exam problem also illuminated a misconception which became evident in other students’ work in addition to Kym’s and Hannah’s: discretizing the Taxicab metric and not operating with it as a continuous measure. This led Kym and Hannah to somewhat disregard their understanding of the preciseness of the definition of a circle to identify locations which were almost equidistant from the buildings, but not exactly. This is consistent with Smith (2013) in that the author found it necessary to have conversations with students about how it was possible to draw line segments “through the grid” even though a car would not be able to drive through the blocks in a city. Referring back to Fischbein (1993) and this idea of a figural constraint creating pressure to neglect a definition, in these cases the figural constraint was the manner in which distance was defined. Thus, when introducing the Taxicab metric, educators should emphasize the continuity of the metric even when illustrating Taxicab concepts with situations that are discrete in real life.

In this paper, illustrations of various understandings of concepts in Taxicab geometry exhibited by two students in a college geometry course were provided. In the given problem, we hoped to help students develop a deeper understanding of these definitions and how to apply them. By using APOS Theory to analyze these students’ solutions to a real-life situation, we were able to uncover some common misconceptions about Taxicab distance and circles. For example, multiple students believed it was not possible to travel in non-integer increments, i.e. – “split” units. This did lead to students attempting to optimize distance under a certain constraint, which was not intentional. These students could imagine a circle with a center that satisfied the problem but struggled to correctly identify a point that would actually be equidistant from the two buildings specified in the problem. We provide a suggestion to add as supplement to existing questions or to re-phrase the initial problem. By doing so, we hope to gather more details about how students understand the concept of **Circle**. This suggestion is as follows: Draw a graphical representation of how a Taxicab circle could be used to identify a location for your apartment, given that (i) You want to be exactly halfway between the buildings, and (ii) you do not want to be exactly halfway between the buildings. Is there more than one way to do each of these? What is this distance called in relation to the definition of a circle? Can you write the equation of either of the circles you have identified in (i) and (ii) using the definition of a circle?

Future research would investigate if these questions would help students to better assimilate the concept of a **Circle** in Taxicab geometry into their existing *circle schema.* There are other concepts that could emerge from the initial question posed such as **Midpoint** and **Perpendicular bisector**. Further research would investigate what questions could be asked for students to better develop their understanding of these concepts as well.

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