Business Calculus Students' Understanding of Marginal Functions

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Business majors represent a significant proportion of the population of students enrolled in calculus at the college level. However, there is a lack in research literature that tackles the teaching and learning of business applications at this level. This pilot study represents the beginning phases of a project that aims to investigate business students' reasoning through tasks pertaining to marginal analysis (derivatives in a business context), accumulation functions and Riemann sums. A preliminary analysis of interviews with two pairs of students is presented, with an emphasis on their thought process while answering questions related to cost, revenue and profit functions as well as their marginal counterparts. The context-based activities were designed with a realistic mathematics education perspective, motivated by guided reinvention.

Keywords: Business Calculus, Marginal, Derivatives, Realistic Mathematics Education.

The teaching and learning of calculus, relative to the theoretical advances in education, is a research area that has only recently gained the interest of mathematicians, educators and psychologists. Researchers such as Warnock, Orton, Tall, Vinner are considered the founders of the aforementioned field, and it was their work in the early 1980s that created the foundations for future research and a need for curricular reform (Rasmussen, Marrongelle, & Borba, 2014). Standard calculus topics such as limits and derivatives have been typically researched over the past couple of decades, but it was not until the mid-2000s that the teaching and learning of Riemann sums and definite integrals became an area of interest for some researchers. When it comes to the teaching and learning of business calculus, there is a scarce amount of research that deals with the cognitive obstacles that students face. In fact, business students represent about 45% of the students that are enrolled in a first semester calculus at the university where this study was conducted. The lack of representation of this population of students in research studies is problematic.

Motivation and Relevant Literature

Originally, the pilot study the authors-researchers had in mind was designed to tackle students' understanding of accumulation and Riemann sums in business contexts. The activity we are considering in this report was intended to prime them on rate of change in a business setting through the context of marginal cost, revenue and profit. We assumed that students came in with the knowledge since they had already covered it in class. When analyzing their work on that introductory task, we found that their understanding of derivatives and marginal cost, revenue and profit was not as fully developed as we had expected. Our research focus thus shifted to analyzing what the participants understood in order to build a new activity centered around marginal functions.

Throughout the interview, students struggled to express their understanding of marginal quantities relatively to the context. This issue seemed to support previous theories that students' struggle with the concept of integration is strongly related to a poor understanding of rates of change (Kouropatov & Dreyfus, 2014; Thompson, 1994; Thompson & Silverman, 2008). Some studies documented the main issues that arise with students' interpretation of derivatives and noted that they generally perform derivative computations without paying close attention to what the values represent (Bressoud, Ghedamsi, Martinez-Luaces, & Törner, 2016). When it comes to studies that analyze how students reason through problems involving applications of derivatives in business contexts, research is very limited. To our knowledge, only Mkhatshwa and Doerr (2016, 2017) first investigated context-based opportunities to learn for business calculus students and then focused on revenue maximization applications. In addition, students primarily view integrals as a tool to calculate areas of unconventional shapes using the antiderivative of the integrand. This is not enough for them to thoroughly understand the multiplicative summation structure and thus utilize it in non-routine situations (Jones, 2015; Sealey, 2006). Therefore, we took this opportunity to first analyze business calculus students' understanding of marginal functions, which would eventually reinforce their understanding of accumulation functions, Riemann sums and definite integrals conceptually rather than algorithmically.

Because this study represents the beginning phases of a bigger project tailored towards the aforementioned topics, our analysis revolved around the following questions:

- How do students interpret and analyze the cost, revenue and profit functions as well as the relationship between them?
- What are some of the observations that can be made with regards to student interpretation of marginal cost, revenue and profit values on optimal business strategies?

Theoretical Perspective

The original tasks, including the one we focus on in this report, were designed under a Realistic Mathematics Education (RME) lens through guided reinvention. RME is a teaching and learning theory developed at the Freudenthal Institute in the Netherlands. Historically, mathematics instruction is typically done through formal definitions, theorems and occasional proofs. Contextual applications are usually given as concluding activities to relate the formal theory to real life examples. RME advocates argue that mathematics should be viewed as a *human activity* (Hough & Gough, 2007). To this end, guided reinvention is utilized so that students engage in their own learning and "reconstruct" the mathematics that they are expected to learn (Freudenthal, 1978; Stephan, Underwood, & Yackel, 2014). In addition, the context and models should be *experientially real* to the students, in the sense that students need to connect what they are doing to the ultimate goal of the lesson (Stephan et al., 2014). According to Treffers (1987), teaching from an RME perspective requires the use of contexts and models, allowing students to construct their own mathematical understanding through interactive learning. Our sequence of tasks was designed to help students reinvent the big ideas within accumulation in a Business Calculus context. Due to the obstacles that appeared during students' interpretation of marginal values, we decided to limit this study to an analysis of the latter topic.

Context wise, marginal cost (or revenue or profit) is the instantaneous rate of change of cost (or revenue or profit) relative to production at a given production level (cite book). Hence, if x represents the quantity of items produced and sold in a hypothetical business context, the marginal revenue $R'(x)$ is the derivative of the revenue function $R(x)$, the marginal cost $C'(x)$ is the derivative of the cost function $C(x)$, and the marginal profit $P'(x)$ is the derivative of the profit function $P(x)$. For instance, a value of $P'(50) = 24$ means that the marginal profit at a production level of 50 items is 24 dollars/item. This implies that if the company produces and sells one additional unit, thus at the sale of the 51^{st} item, it is expected to gain about \$24. The same reasoning applies to marginal cost (estimate for the cost of production of the $(n + 1)^{st}$ item) and marginal revenue (estimate for the revenue generated from the sale of the $(n + 1)^{st}$ item) at a production level of n items.

Methods

During a summer semester at a large university, students from a business calculus course were given the opportunity to participate in a 90-minute recorded session while they work through all the tasks that were originally designed, and answered some questions asked by the researcher. The task that is the focus of our report situated students in a hypothetical jacket manufacturing company. Given the fixed and variable costs of production, as well as a quadratic revenue function, they were asked to develop a model for both the cost and revenue functions, as well as the corresponding marginal functions. Then, students had to evaluate those functions and their marginals at two different levels of production in order to decide whether or not it would be a good business move for the company to produce that many jackets. The last question prompted students to find the production level that would maximize the company's profit.

Two pairs of students volunteered to participate. They were split in two groups: Piper and Jay (Group 1), Mo and Ty (Group 2). The students had diverse ethnic backgrounds (African American, Hispanic and white). The researcher allowed students to get comfortable working together first and limited his interaction with them until it was clear that they were collaborating and sharing ideas. For both pairs, the first task (the one we are analyzing in this report) took about 45 minutes to complete, which included discussion time with the researcher. Hence, around 90 minutes of audio-video footage were analyzed using an open coding process in which annotations and comments were split into the two major themes that are elaborated in the next section. Below is a reproduction of the questions presented to the students in this first task.

A company manufactures jackets. The costs of rent and utilities are fixed at $2800 per month, and each jacket costs $4 to produce. In addition, suppose that the revenue function is given by $R(x) = 320x - 0.08x^2$ where x represents the number of jackets produced monthly.

a) Find the cost function and the profit function.
b) What is the cost of producing 1000 jackets monthly? Find the corresponding revenue and profit.
c) Do you think it is a good idea for the company to produce 1000 jackets monthly? Elaborate.
d) What is the cost of producing 3000 jackets monthly? Find the corresponding revenue and profit.
e) Do you think it is a good idea for the company to produce 3000 jackets monthly? Elaborate.
f) Find the marginal cost, marginal revenue and marginal profit functions.
g) Evaluate the marginal cost, revenue and profit when producing 1000 jackets. Interpret your answers.
h) Evaluate the marginal cost, revenue and profit when producing 3000 jackets. Interpret your answers.
i) Are you still certain from your answers to parts c) and e)? Explain.
j) How many jackets do you think the company should produce and sell in order to maximize profit?

Figure 1. Questions from the first task of the interview.

Preliminary Findings

After a careful examination of the conversations between the participants, as well as the participants and the researcher, it seemed like students' have uncertainties in how the cost, revenue and profit functions are related as well as what are the implications of the marginal function values at specific levels of production.

Analysis and Interpretation of Business Functions

The first half of the task tackled students' familiarities with functions that model the cost, revenue and profit. Given fixed costs for production and variable unit prices, students were asked to model a cost function and were expected to utilize a linear model. After some guidance from the researcher, group 1 correctly modeled the cost function. When asked to write an expression for the profit function, Piper noted that "profit equals revenue minus cost because revenue is more, and cost is less", which may indicate she believes that profit always represents a positive quantity, a gain. Given a production level of 1000 jackets where profit is positive, the students were asked if it is a good idea for the company to produce that many jackets. Students in this group thought it was indeed beneficial because "the profit is substantially greater than the initial cost", seemingly thinking of those as two comparable quantities. When asked to find the maximum profit, Piper and Jay utilized Desmos to plot the graph of the profit function and were able to locate the maximal value. Students in group 2 also compared cost to profit at a production level of 1000 jackets. Remarkably, when production level changed to 3000 jackets, they compared the cost and profit at that level to those at 1000 jackets "[the company ends] up making the same amount of money, but it costs more to produce". When asked to find the maximum profit, Mo and Ty started by equating the revenue to the cost function. This reflects a confusion between the concept at hand and the break-even points, where the profit is in fact null. After discussing this idea with them, they took a graphical route. Surprisingly, they plotted the cost and revenue functions, but not the profit function and tried to estimate the production level (x value) that "maximizes the distance between the cost and the revenue […] biggest positive difference since you can have a bigger difference down there [pointing at regions of loss] but then [the company] would be losing money." Their analysis of the difference between revenue and cost in lieu of analyzing the profit function directly indicates they may not have a robust understanding of the relationship between the three quantities.

Analysis and Interpretation of Marginal Functions

The second half of the task prompted students to answer questions pertaining to marginal cost, revenue and functions. Both groups had no issues in connecting marginals to derivatives and were able to find them using the power rule for polynomials. Group 1 was also familiar with the linearity property of the derivative, since they subtracted the marginal cost from the marginal revenue as a shortcut to finding the marginal profit. However, all four students seemed to face some obstacles when asked to interpret the marginal values they obtained at given levels of production. The hindrances started during a conversation about what "derivative" means to them in any context. One particular answer reflected students' association of derivatives with an algorithmic process without paying close attention to its connection with rates of change: "it's kind of like taking the original form and then transforming it or condensing it into something else". In addition, For the given levels of production, students struggled to utilize correct terminology while interpreting the marginal values they obtained. For instance, Piper noted that "at [a production level of] 1000 jackets, the ROC for profit is 156, and there is no ROC for the cost". Besides not using units with their values, it seemed like a constant rate of change for a function was mistaken with the function itself being constant, thus having a null rate of change.

Perhaps the most notable observation, after redirecting the students to derivative values being slopes of tangent lines, is that students used linear approximations of functions to approximate the additional revenue and profit for any other level of production. The excerpt below showed us elements of a productive understanding of liner approximations and how they relate to rates of change, but the robustness that is needed for a more advanced interpretation was yet to be developed. While the reasoning below applies to the case of linear cost functions (the marginal cost is constant thus each additional unit costs the same to produce), it cannot be extended to the case of marginal revenue and marginal profit. The following is an excerpt of Mo's interpretation for the marginal values he found at a production level of 1000 jackets.

Mo: So, for marginal cost, it's like per unit or whatever so for cost, it's going to be each unit costs $4 to produce on top of the last one. And then same for [the revenue and profit], each unit nets us $160 more revenue than the last one we made, and each unit gives us a profit $156 more than the last one. *(Ty agreed with Mo's statement)*

Interviewer: What do you mean by "each unit"?

Ty: Like each additional unit

Interviewer: So, if I'm at 1000 [units] and I [produce] one more unit, then […] the revenue is going to be $160. If I produce 10 more units after, is my revenue going to be $1600?

Mo: I think that's the assumption that it will do that as long as you're basing it on that 1000

Interviewer: Your starting point is 1000 [jackets] and after that you can take any value […] and estimate the additional revenue?

Mo: Yes, so like when we base it on 3000 [jackets] now we have a negative number because we're starting to lose money on each one that we make additionally.

Interviewer: Okay so your "each" means that starting with 1000 or 3000, you can [increase] by as much as you want, say by 10, 100, 500 units… and then that would tell you what your additional cost, revenue and profit are?

Ty: Yea for each one, I guess.

This presents evidence to support our next claim that Mo and Ty understand the effect of a marginal value's sign (positive or negative) on gain or loss. However, the use those values as *local* approximations of the additional revenue or profit is a skill that is yet to be acquired.

Implications

Our preliminary findings suggest that students did not master additional preparation in order to give correct interpretations of the marginal cost, revenue and profit functions. Finding derivatives by hand is a skill that is typically focused on during traditional calculus courses but with all the software available to do that in practical applications, it would be more beneficial for them to demonstrate strong analytical skills through interpreting the meaning of the marginal values. Taking our observations in this study into consideration, our next step would be to conduct a teaching experiment that emphasize on ideas that are not typically focused on, such as the profit being a quantity that could be positive or negative, the revenue and cost being comparable and how they relate to profit, as well as using correct vocabulary and units to describe the meaning of the derivative in any context. Thus, our ensuing goal is to create a sequence of tasks that guide students through the theoretical underpinnings of business functions and their marginal counterparts as bases for optimal business strategies. We have posited some initial learning goals for the next tasks that will be tailored towards students being able to:

1. Write and evaluate the cost, revenue and profit functions using given information (fixed and variable costs, price-demand equation…)
2. Interpret and analyze the cost, revenue and profit functions at a given level of production
3. Derive the marginal cost, revenue and profit functions.
4. Interpret and analyze the marginal cost, revenue and profit at a given level of production.

References

Bressoud, D., Ghedamsi, I., Martinez-Luaces, V., & Törner, G. (2016). *Teaching and Learning of Calculus*.

Freudenthal, H. (1978). Address to the first conference of I.G.P.M.E. (International Group for the Psychology of Mathematical Education), at Utrecht 29 August 1977. *Educational Studies in Mathematics*, *9*(1), 1–5.

Hough, S., & Gough, S. (2007). Realistic Mathematics Education. *Mathematics Teaching*, (July).

Jones, S. R. (2015). The prevalence of area-under-a-curve and anti-derivative conceptions over Riemann sum-based conceptions in students' explanations of definite integrals. *International Journal of Mathematical Education in Science and Technology*, *46*(5), 721–736.

Kouropatov, A., & Dreyfus, T. (2014). Learning the integral concept by constructing knowledge about accumulation. *ZDM - International Journal on Mathematics Education*, *46*(4), 533–548.

Mkhatshwa, T., & Doerr, H. M. (2017). Students' quantitative reasoning in a revenue maximization context. *Conference on Research in Undergraduate Mathematics Education, the 20th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp. 755-762) . San Diego, California.

Mkhatshwa, T., & Doerr, H. M. (2016). Opportunity to learn solving context-based tasks provided by business calculus textbooks: An exploratory study. In T. Fukawa-Connelly, N. Infante, M. Wawro, & S. Brown (Eds.), *Proceedings of the 19th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp. 1124-1132). Pittsburgh, Pennsylvania.

Rasmussen, C., Marrongelle, K., & Borba, M. C. (2014). Research on calculus: what do we know and where do we need to go? *ZDM - International Journal on Mathematics Education*, *46*(4), 507–515.

Sealey, V. (2006). Definite integrals, Riemann sums, and area under a curve: What is necessary and sufficient. *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, *2*(1991), 46–53.

Stephan, M., Underwood, D., & Yackel, E. (2014). Guided Reinvention: What Is It and How Do Teachers Learn This Teaching Approach?, (April 2016).

Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, *26*(2–3), 229–274.

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. *Making the Connection: Research and Teaching in Undergraduate Mathematics Education*, 43–52.

Treffers, A. (1987). Three Dimensions: a Model of Goal and Theory Description in Mathematics Instruction - The Wiskobas Project. Dordrecht: Springer Netherlands.