Generalizing Actions of Forming: Identifying Patterns and Relationships Between Quantities

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In this paper, we illustrate and discuss two undergraduate students' reasoning about quantities' magnitudes. One student identified regularities regarding the relationship between two quantities by focusing on successive amounts of change of one quantity (i.e., a pattern) while the other attended to relative amounts of changes in both quantities (i.e., a relationship). We illustrate that although reasoning about amounts of change is useful for making sense of the rate of change in quantities, reasoning about relative changes in identifying a relationship between quantities' magnitudes is likely more productive in developing the concept of rate of change.

Keywords: Quantitative and covariational reasoning, Generalization, Rate of change.

Quantitative and covariational reasoning is critical to supporting students in understanding major pre-calculus and calculus ideas (Ellis, 2007b; Confrey & Smith, 1995; Thompson, 1994, 2011; Thompson & Carlson, 2017). Moreover, Ellis (2007b) reported that quantitative reasoning plays a significant role in students' constructing productive generalizations. In this paper, we characterize two undergraduate students' generalizing actions during a teaching experiment focused on modeling covariational relationships. We give specific attention to how the students' engagement in covariational/quantitative reasoning differed and, in turn, how this difference led them to generalize different regularities regarding a covariational relationship between two quantities' magnitudes. We report the generalizing actions of two students, with one student operating with additive comparisons of amounts of change in one quantity, and the other student operating with additive and multiplicative comparisons of amounts of change of two quantities (i.e., relative changes and ratios). We also report the resulting identified regularities of these ways of operating.

Background and Theoretical Framework

Quantitative and Covariational Reasoning

This study focuses on students' generalizing actions involved in reasoning with relationships between quantities in dynamic situations. We use *quantity* to refer to a conceptual entity an individual construct as a measurable attribute of an object (Thompson, 2011). We also describe students' construction of *quantitative structures* by characterizing their *quantitative operations* when determining a *quantitative relationship*. By *quantitative operation*, we mean the conception of producing a new quantity from two others, and by a *quantitative structure*, we mean a network of *quantitative relationships* (i.e., the conception of these three quantities; Thompson, 1990, 2011). For example, someone can create a quantity as a result of *additive comparison* of two quantities by answering the question, "How much more (less) of this is there than that?", whereas someone can create a quantity as a result of *multiplicative comparison* of two quantities by answering the questions "'How many times bigger is this than that?' and 'This is (multiplicatively) what part of that?'" (Thompson, 1990, p. 11).

Furthermore, when students engage in a dynamic context that involve two quantities varying simultaneously, they need to coordinate quantitative operations with *covariational reasoning* (i.e., attending to how one quantity varies in relation to the other in tandem; Saldanha &

Thompson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). For example, in order to determine a pattern of *differences* in a quantity's variation in relation to the other, a student can coordinate the variation of two quantities values or magnitudes and the variation of the resultant *difference* quantity's values or magnitudes (e.g., as two quantities increase, the difference of these quantities decrease; see Mental Action 3 in Carlson et al., 2002).

Non-Ratio and Ratio-Based Reasoning

Many researchers have provided different ways of making sense of rate of change of one quantity with respect to another. For example, some researchers (e.g., Carlson et al., 2002; Confrey and Smith, 1994, 1995; Ellis, 2007b, 2011; Johnson, 2012, 2015b; Liang & Moore, 2017, 2018; Monk & Nemirovsky, 1994; Tasova & Moore, 2018) argued the importance of non-ratio based reasoning, which is reasoning about amounts of change in one quantity in relation to uniform changes in another quantity. For example, a constant rate of change in the perimeter of a square with respect to changes in side length can be conceived by determining that amounts of increase in the perimeter is "two centimeters" each time "if you increase both sides by point five [centimeters]" (Johnson, 2012, p. 322).

There are also researchers (e.g., Confrey and Smith, 1994, 1995; Ellis, 2007b, 2007c, 2011; Ellis, Ozgur, Kulow, Williams, & Amidon, 2013, 2015; Johnson, 2015a) who have argued for the importance of ratio-based reasoning (i.e., forming ratios of one quantity's change to the other quantity's change) in making sense of the rate of change. For example, a constant speed of a Clown can be conceived as a ratio of distance to time (i.e., "5cm:4s"; Ellis, 2007b, p. 472). We note that these conceptualizations (mostly) included students' reasoning with numbers. In this paper, we expanded this body of literature by demonstrating ways in which students make sense of rate of change in dynamic events and in graphs by reasoning with quantities' *magnitudes* independent of numerical values (see Liang, Stevens, Tasova, and Moore [2018] and Thompson, Carlson, Byerley, and Hatfield [2014] for a detailed discussion on *magnitude reasoning*). Because reasoning with quantities' magnitudes does not necessitate reasoning with specified values of the quantities, we conceptualize "ratio-based reasoning" as reasoning with a "quotient [that] entails a multiplicative comparison of two quantities with the intention of determining their relative size" (Byerley and Thompson, 2017, p. 173). We aim at demonstrating students' generalizing actions by characterizing how they operate with magnitudes within a complex quantitative structure.

Generalizing Framework

Building on Ellis' (2007a) taxonomy of generalizations, Ellis, Tillema, Lockwood, and Moore (submitted) introduced a generalization framework involving three major forms of students' generalizing—relating, forming, and extending. Students' generalizing actions of *forming* occur within one context, task, or situation. This type of generalizing action includes students *searching for similarity and regularity* across cases, *isolating constancy* across varying features by *establishing a way of operating* that has the potential to be repeated, and *identifying a regularity* across cases, numbers, or figures. In this paper, we are using this framework to illustrate two students' generalizing actions of forming by focusing on their establishing ways of operating and identifying regularities as they relate to covarying quantities.

Method

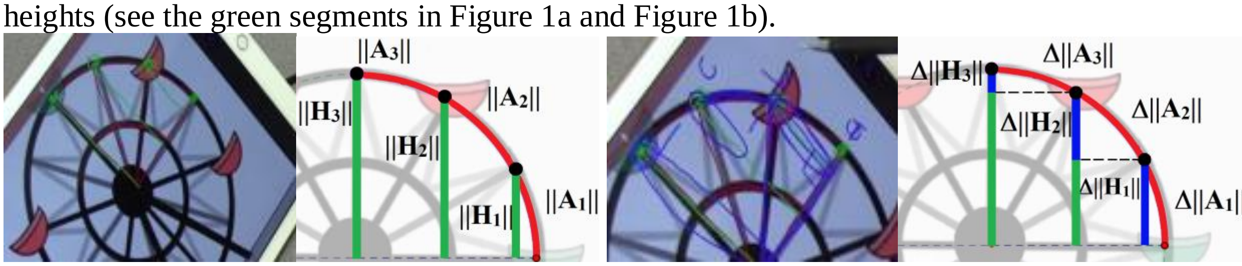
The data we present in this paper is from two semester-long teaching experiments (Steffe & Thompson, 2000) conducted at a large public university in the southeastern U.S. A common goal of both teaching experiments was to investigate undergraduate students' mental actions involved in reasoning with dynamic situations, magnitudes, and graphs from a quantitative and covariational reasoning perspective. In this paper, we focus on a student, Lydia, who at the time of the study, was a pre-service secondary mathematics teacher in her first year in the program, and another student, Caleb, who was a sophomore majoring in music education. Lydia participated in 11 videotaped teaching experiment sessions and Caleb participated in 14, each of which was approximately 1–2 hours long. We transcribed the video and digitized these students' written work for both on-going and retrospective conceptual analyses (Thompson, 2008) to analyze their observable and audible behaviors (e.g., talk, gestures, and task responses) and to develop working models of their thinking. We choose to present these two cases here because the students' generalizing actions including their established ways of operating and identified regularities are cognitively distinct, and thus are worth documenting and contrasting.

Analysis and Findings

In this paper, we illustrated two students' generalizing actions—by focusing on their *ways of operating* and *identified regularity* as they determined the covariational relationship between two quantities.

Lydia's Generalizing Actions

First, we characterize Lydia's activities in Taking a Ride to discuss her generalizing actions of *establishing a way of operating* (see Tasova & Moore [2018] for detailed account of her generalizing activity). To start with, we presented Lydia an animation of a Ferris Wheel rider that was indicated by a green bucket rotating counterclockwise from the 3:00 position (Desmos, 2014). Then, we asked her to describe how the height of the rider above the horizontal diameter changes in relation to arc length it has traveled. After reasoning about directional change in height in relation to arc length (i.e., height is increasing as the arc length increases in the first quarter of rotation), she engaged in *partitioning activity* (Liang & Moore, 2017, 2018) in order to investigate how height changes in relation to arc length. Namely, she used the spokes of the Ferris wheel (i.e., each of the black bars [see Figure 1a] connecting the center of the wheel to its edge) to partition the Ferris wheel into equal arc lengths, and then she drew corresponding heights (see the green segments in Figure 1a and Figure 1b).



(a) (b) (c) (d)

Figure 1. Lydia engaging the Taking a Ride task. Figure 1b and 1d were designed for the reader.

With support from the teacher-researcher's (TR) questioning, Lydia constructed successive amounts of change in height (i.e., circled in blue seen in Figure 1c and blue segments in Figure 1d) that corresponded to successive uniform incremental changes in arc length. That is, Lydia *established a way of operating* that involved the construction of a new quantity (i.e., amounts of change in height) and associated partitioning activity. We inferred from her activity that Lydia was constructing the *difference* of every two consecutive height magnitudes (i.e., $\Delta\|H_1\|$, $\Delta\|H_2\|$, and $\Delta\|H_3\|$, see blue segments in Figure 1d) corresponding to the magnitude of arc length that accumulates in equal increments; Smith III & Thompson, 2008; Thompson, 1990). This served as evidence that she was operating with additive comparisons among the accumulated height magnitudes at successive states (i.e., $\|H_1\|$, $\|H_2\|$, and $\|H_3\|$, see green segments in Figure 1b)). What's more, she additively compared the amounts of change magnitudes in height. Namely, she concluded $\Delta\|H_1\| < \Delta\|H_2 < \Delta\|H_3\|$.

We note that in her additive comparison, Lydia was not interested in measuring how much one quantity's magnitude exceeded (or fell short) of another quantity's magnitude. Instead, her quantitative operation included a gross additive comparison (Steffe, 1991) between the amounts of change within a quantity (e.g., $\Delta\|H_3\|$ being "smaller" than $\Delta\|H_2\|$). From this activity, therefore, we inferred that Lydia made a gross comparison of the differences, which is a more complex quantitative reasoning because this requires relating results of quantitative operations (i.e., an additive comparison of the results of two additive comparisons). After engaging in repeated additive comparisons, Lydia was able to *search* for pattern in those quantities' variation. With the recognition in the pattern of differences (i.e., decreasing change in height along with those equal partitioning in arc length as shown in Figure 1c and 1d), Lydia had *identified the regularity* in how height's magnitude changes in relation to arc length in the first quadrant, stating "as the arc length is increasing... [the] vertical distance from the center is increasing ... but the value that we're increasing by is decreasing."

Caleb's Generalizing Actions

We demonstrate Caleb's generalizing actions when engaging in the Changing Bars Task, which involved a simplified version of Ferris wheel situation (i.e., a circle) and six pairs of orthogonally oriented bars (see Figure 2). On the circle, the red segment represents the magnitude of the riders' height above the horizontal diameter and the blue segment represents the magnitude of the rider's arc length traveled from the 3 o'clock position. Caleb was able to move the end-point (i.e., the rider) along the circle between the 3:00 position to the 12:00 position. We asked Caleb to choose which, if any, of the orthogonal pairs accurately represents the relationship between the height and the arc length of the rider as it travels.

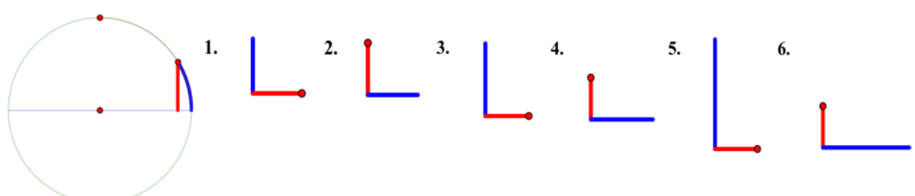
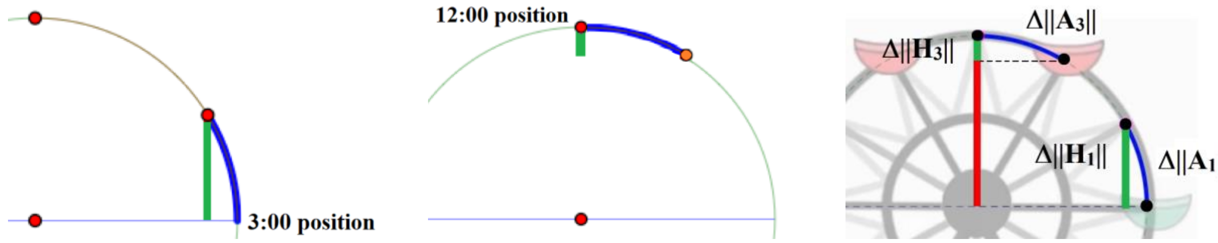


Figure 2. *Changing Bars Task (numbering and locations of the six pairs was edited for readers).*

In this section, we report Caleb's generalizing actions that involved him establishing ways of operating that entailed additive *and* multiplicative comparisons. We note that identifying these different operations does not imply that Caleb engaged in them in order. We believe that Caleb's reasoning involving additive and multiplicative comparisons was internally coherent and he could make claims about either one depending on the TR's questioning. Our goal of making such distinction was to characterize his different ways of operating and contrast his ways of operating with those of Lydia.

Additive comparison of amounts of change. Caleb started with comparing the amounts of change in arc and amounts of change in height as the dynamic point traveled a small distance from the 3:00 position. He stated that, "...at the very beginning, ... the height above the center and the distance traveled from 3:00 position should be similar." This *way of operating* was repeated several times during his generalizing actions with use of slightly different verbal statements. For example, in a later conversation, he stated “at the beginning of the path [*referring to 3:00, see Figure 3a*], ...the rate at which the height increases should be almost equal to the rate at which the distance it's traveled.” We note that although he used the word of “rate,” we infer that he meant amount of change in height and arc length. By repeating the same way of operating in 12:00 position (i.e., new case in the first quarter of rotation), Caleb further stated that:

> ...from this point [*pointing to the point denoted in orange in Figure 3b]* ... to this point *[pointing to 12:00 position in Figure 3b*], the height barely changes [*green segment in Figure 3b and Figure 3c (i.e.,* $\Delta\|H_3\|$*)*], but you’re still traveling a fair distance around the circle [*blue annotation in Figure 3b and blue segment (i.e.,* $\Delta\|A_3\|$*) in Figure 3c*].



(a) (b) (c)

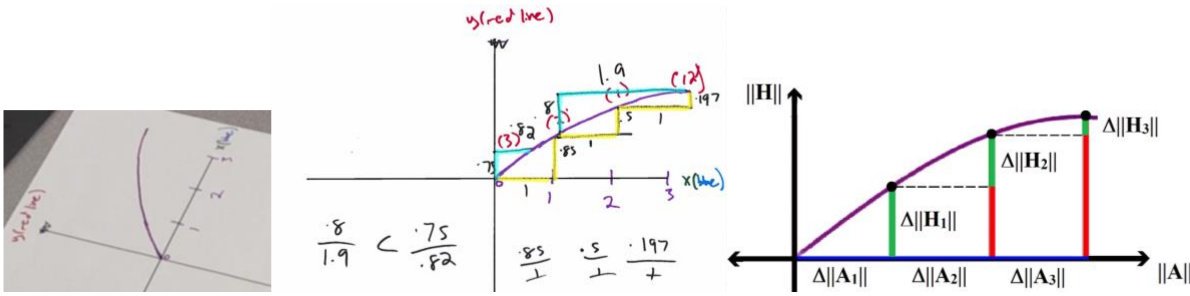
Figure 3. Recreation of Caleb’s activity in the Changing Bars task.

From his activity, we infer that Caleb’s established way of operating included an additive comparison of $\Delta\|H_1\|$ with $\Delta\|A_1\|$ near the 3:00 position (i.e., $\Delta\|H_1\|$ is almost equal to $\Delta\|A_1\|$) and of $\Delta\|H_3\|$ with $\Delta\|A_3\|$ near the 12:00 positon (i.e., $\Delta\|H_3\|$ is smaller than $\Delta\|A_3\|$). Similar to the case of Lydia, we did not have evidence that Caleb constructed the difference between amounts of change in two quantities (e.g., how much $\Delta\|H_3\|$ exceeded of $\Delta\|A_3\|$) beyond a gross additive comparison between the amounts of change in each quantity (Steffe, 1991).

As the teaching experiment proceeded, he *isolated a constant feature* of the relationship between the amounts of change in height’s magnitude and the amounts of change in arc length’s magnitude across the first quarter of rotation. He stated that “from any point to any other point along this stretch [*referring to the first quarter of rotation*], the amount that the red line [i.e., height’s magnitude] changes should always be smaller than the amount that the blue line [i.e., arc length’s magnitude] changes.” Therefore, we infer that Caleb *isolated a constant feature* across varying features of the relationship between $\Delta\|H\|$ with $\Delta\|A\|$ without reaching the final stage of fully describing an identified regularity across the first quarter of rotation (e.g., $\Delta\|H\|$ becomes smaller relative to $\Delta\|A\|$ as the rider travels from 3:00 positon to 12:00 position). It is important to note that, however, Caleb knew that “when we’re looking down here [*refers to 3:00 position*]” the relationship between $\Delta\|H_1\|$ and $\Delta\|A_1\|$ “should be vastly different from” the relationship between $\Delta\|H_3\|$ and $\Delta\|A_3\|$ (see Figure 3c).

Eventually, Caleb *identified a regularity* regarding the relationship between $\Delta\|H\|$ and $\Delta\|A\|$ across the all cases in the first quarter of rotation. He stated that “the further you move away from the 3:00 position, the more variance there would be between the red (i.e., $\Delta\|H\|$) and the blue lines (i.e., $\Delta\|A\|$)” and by “variance” he meant that $\Delta\|A\|$ became much bigger than $\Delta\|A\|$ as the dynamic point approached the 12:00 position.

Multiplicative comparison of amounts of change. Caleb also *established a way of operating* that involved multiplicative comparisons between $\Delta\|H\|$ and $\Delta\|A\|$. He stated that “As we approach this point right here [*refers to 12:00 position*], the ratio of the rate at which the height increases to the rate or to the distance we've traveled around...the circle, um, is at its smallest..." This way of operating was also repeated several times during his generalizing activity—both in the circle situation and in six pairs of bars. For example, near 12:00 position, he established that there is a "...1 to 2.5 or 1 to 3 ratio in the amount that you change the red line's length [i.e., height's magnitude] decreases to the blue line length [i.e., arc length's magnitude] decreasing." We infer that Caleb constructed a quantity as ratios of $\Delta\|H\|$ to $\Delta\|A\|$ across the first quarter rotation, and anticipated that the ratio gets smaller as the rider travels from 3:00 position to 12:00 position. His way of operating that entailed multiplicative comparisons of quantities and his identified regularity regarding the relationship between height and arc length became evident in his graphing activity, which we report next.



(a) (b) (c)

Figure 4. (a) Caleb's initial graph, (b) a resulting drawing of Caleb's partitioning activity, and (c) a recreation of Figure 4b for readers

Caleb's graphing activity. The researchers then asked Caleb to produce a graph that represents the relationship between height and arc length. He constructed the concave down graph shown in Figure 4a. To interpret his displayed graph in terms of amounts of change in height and arc length, Caleb engaged in partitioning activity (Liang & Moore, 2017, 2018) to construct incremental changes that represented amounts of change in height (i.e., $\Delta\|H_1\|$, $\Delta\|H_2\|$, and $\Delta\|H_3\|$ in Figure 4c; also see yellow vertical segments in Figure 4b) in relation to uniform changes in arc length (i.e., $\Delta\|A_1\|$, $\Delta\|A_2\|$, and $\Delta\|A_3\|$ in Figure 4c; also see yellow horizontal segments in Figure 4b). He then assigned estimated values for each segment to indicate its magnitude (i.e., $\Delta\|A_1\| = \Delta\|A_2\| = \Delta\|A_3\| = 1$; $\Delta\|H_1\| = .85 > \Delta\|H_2\| = .5 > \Delta\|H_3\| = .197$ and constructed ratios of each corresponding pairs, writing ".85/1", ".5/1", and ".197/1" (see Figure 4b). Caleb also operated on these ratios by additively comparing them, anticipating that these ratios should decrease—".85/1>.5/1>.197/1". This suggested that Caleb continued and generalized his ways of operating in the circle and bar situation to the graphical contexts and identified a regularity that the ratios of successive pairs of amounts of change in height and arc length should decrease as the rider travels in the first quarter of rotation. Caleb was also able to extend his ways of operating to non-uniform intervals. Namely, he anticipated that when increments of arc length are not equal (see his partitions in light blue in Figure 4b and his estimated values for each increment), the same regularity should hold, writing ".8/1.9<.75/.82" (see Figure 4b) without calculating the resulting value of the ratios.

Discussion

We focus on two students' generalizing actions by giving attention to their ways of operating and identified regularity. In establishing ways of operating, both Lydia and Caleb first constructed *differences (i.e., amounts of change)* in magnitudes of height (i.e., $\Delta\|H_1\|$, $\Delta\|H_2\|$, and

$\Delta\|H_3\|$) in relation to change of arc length's magnitude (i.e., $\Delta\|A_1\|$, $\Delta\|A_2\|$, and $\Delta\|A_3\|$). However, the way they operated on those *differences* differed. For example, they both engaged in quantitative operation of additive comparisons; however, the operands that they considered in their quantitative operations were different. That is, Lydia additively compared the successive amounts of change in height's magnitude, whereas Caleb additively compared amounts of change in height's magnitude with the corresponding amounts of change in arc length's magnitude. Therefore, the operands for Caleb were *differences* in height and *differences* in arc length (i.e., $\Delta\|H_n\|$ and $\Delta\|A_n\|$, where n=1, 2, and 3), as opposed to Lydia whose operands were *differences* of height in successive states (i.e., $\Delta\|H_n\|$ and $\Delta\|H_{n+1}\|$, where n=1 and 2). We conjecture that the way of additive comparison in Caleb's case might be more productive for generalizing the rate of change in height with respect to arc length since such comparison afforded him to anticipate a resultant ratio of *differences* in height and *differences* in arc length (i.e., $\Delta\|H_n\|/\Delta\|A_n\|$).

We also find that these two students' different ways of operating led them to identify different regularities regarding the similar situations. Lydia searched for the *pattern* (Ellis, 2007b) by making within-measure additive comparisons among heights in different states. Thus, she identified a pattern of how amounts of change in the height decrease as the arc length increases. Caleb searched for the *relationship* (Ellis, 2007b) by making between-measure multiplicative comparisons between height's magnitudes and arc length's magnitudes. Thus, he identified a regularity of relative change of the height with respect to the arc length decreases as the rider travels. We conjecture that this way of operating (i.e., multiplicative comparison between changing quantities' magnitudes) and the resultant identified regularity may afford students to develop productive understandings of rate of change.

We want to point out that, when additively comparing the ratios in justifying his identified regularity, Caleb's engagement with numbers does not imply that he performed arithmetic operations in a sense that he wanted to evaluate the quantities' values. We infer that the reason he assigned numbers to quantities' magnitudes is that he needed to "propagate information" (Thompson, 2011, p. 43) in order to deal with the complex quantitative situations. Thompson (2011) claimed that propagation can be made under the conditions of being *aware* of (i) quantitative structure and (ii) "numerical operations to perform to evaluate a quantity in that structure" (p. 43). Even though Caleb did not perform numerical operations to evaluate quantities, he satisfied the conditions of propagation. That is, he used numbers as intuitive measurements of quantities' magnitudes and he was aware of the quantitative structure. Moreover, Caleb's uses of numbers were necessary for him in order to compare the relative size of two quantities' magnitudes. Part of this necessity comes from the fact that there was no way for him to visually represent the magnitude of a quantitative ratio. That is, Caleb used estimated numbers to reason about the relationship between magnitudes of hard-to-visualize quantities, and then re-interpreted this relationship between *values* in the context of quantitative structure in order to propagate information about the relationship between quantities' *magnitudes*. To confirm if this is the case or to characterize the nature of this reasoning, we believe that future research is necessary.

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