An Analysis of a Mathematician's Reflections on Teaching Eigenvalues and Eigenvectors: Moving Between Embodied, Symbolic and Formal Worlds of Mathematical Thinking

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In this paper, we analyzed a mathematician's daily teaching journals of a 5-day series of teaching episodes on eigenvalues and eigenvectors in a first-year linear algebra course. We employed Tall's (2013) three world model, in conjunction with Tall and Vinner's (1981) concept images and concept definitions, to follow the mathematician and instructor's movements between Tall's worlds. The study showed that the instructor strived to build concept images that, while perhaps mirroring his own concept images, did not resonate with the students.

Keywords: Tall's Worlds, Concept images, IOLA, Reflections, Eigenvalues and Eigenvectors

Theoretical Background

How do mathematicians motivate mathematics concepts in teaching? As learners of mathematics, our past experiences bring to mind a variety of teaching styles. There were lectures where the professor only wrote definitions, theorems, and proofs on the board, followed by a number of examples, and in some rare occasions, professors motivated the idea with pictures. Building on Tall and Vinner's (1981) notions of *concept images* and *concept definitions*, Vinner (1991) claimed, "We assume that to acquire a concept means to form a concept image for it. To know by heart a concept definition does not guarantee understanding of the concept. To understand, so we believe, means to have a concept image" (p. 69). Vinner also examined the role of the definition; in his view, definitions help us to form a concept image. However, he noted that "the moment the image is formed, the definition becomes dispensable. It will remain inactive or even be forgotten when handling statements about the concept in consideration" (p. 69). Using the scaffolding metaphor, he compared this idea with building, saying "the moment a construction of a building is finished, the scaffolding is taken away" (p. 69).

Developing these two notions further, Tall's (2010; 2013) three worlds framework for mathematical thinking (embodied, symbolic, and formal) endeavors to lay out the individual mathematics learning journey from childhood to a research mathematician. According to Tall (2010), the embodied world is based on "our operation as biological creatures, with gestures that convey meaning, perception of objects that recognize properties and patterns… and other forms of figures and diagrams" (p. 22). In other words, the various ways of thinking in the embodied world can also be characterized as giving body to an abstract idea. In Tall's (2010, p. 22) words, "The world of operational symbolism involves practicing sequences of actions until we can perform them accurately with little conscious effort. It develops beyond the learning of procedures to carry out a given process (such as counting) to the concept created by that process (such as number)". Finally, Tall defines thinking in the formal world as that which "builds from lists of axioms expressed formally through sequences of theorems proved deductively with the intention of building a coherent formal knowledge structure" (p. 22).

Using Tall's model, Stewart, Thompson, and Brady (2017) investigated a mathematician's (and co-author) movements between Tall's worlds while teaching algebraic topology. In this study, the instructor reported that students experienced the most difficulty in moving from the embodied world into the formal world. Believing the struggle would stimulate mathematical growth in his students, this instructor "refused to give students proofs that were pre-packaged. More specifically, he wanted to provide students with intuitions and pictures that would help them understand the conceptual nature of the proof and ultimately lead them to it" (p. 2262). Stewart (2018) created a set of linear algebra tasks designed to help students move between Tall's worlds. Stewart, Troup, and Plaxco (2018) examined a mathematics educator's (and co-author's) movements as well as decision-making moments while teaching linear algebra. All these studies indicate that movements between Tall's worlds are a rich research topic worthy of ongoing investigation.

As part of the first author's research program, the overarching goal of this study was to examine a mathematician's (the instructor and co-author) movements between Tall's worlds. Throughout this investigation, the instructor often emphasized that his goal was to reach the eigenvalues and eigenvectors section of the course, which motivated the focus of this paper. Furthermore, although research on students' difficulties and understanding of eigenvalues and eigenvectors has increased (e.g., Caglayan, 2015; Gol Tabaghi & Sinclair, 2013; Salgado & Trigueros, 2015; Thomas & Stewart, 2011), research on mathematician's voices and what goes on in the mind of the working mathematician while teaching the eigentheory is still scarce. The data analyzed in this study is from the instructor's reflections on teaching eigenvalues and eigenvectors. Most researchers maintain that reflection is an essential part of teaching mathematics (e.g., Davis, 2006; Fund, 2010; Moore-Russo & Wilsey, 2014;). According to Dewey (1933), reflection is "active, persistent, and careful consideration of any belief or form of knowledge in the light of the grounds that support it and the further conclusions to which it tends" (p. 9). Fund (2010) adds that "teachers need to develop particular skills, such as observation and reasoning, in order to reflect effectively and should have qualities such as open-mindedness and responsibility" (p. 680).

The research questions guiding this study were: (a) How did a working mathematician convey to students a concept image of eigenvalues and eigenvectors? (b) What were some of the factors causing the instructor to move between Tall's worlds?

Methods

This qualitative narrative study (Creswell, 2013) is the second in a series of studies intended to examine the linear algebra instructor's mathematical thought processes while teaching a first course in linear algebra, as well as how they leverage Tall's (2013) three worlds. This study took place over the course of a semester at a Southwestern research university in the US. The analysis focuses on an instructor's observations, as recorded through journal entries, over a five-day period, while implementing tasks from the Inquiry-Oriented Linear Algebra (IOLA) curriculum (Wawro et al., 2013). The research team consisted of a mathematician specializing in differential geometry (the instructor, postdoctoral fellow, and co-author), two mathematics educators, and an undergraduate research assistant student.

Throughout the semester, the instructor recorded his observations on how his class reacted to a variety of teaching styles and ideas. He additionally met with the research team once a week throughout the semester and the following summer to discuss these experiences and reflections. This allowed the researchers to triangulate data via member checking with the instructor directly and additionally afforded him ample time to share a wide variety of teaching experiences, as well as his reasoning and thought processes while making these decisions. To collect additional data on the instructor's teaching from the student's perspective, the research team administered a survey, given as a worksheet, and conducted a student interview.

The research team converted the instructor's journal and the worksheet results into Excel spreadsheets to expedite coding and sorting the data to search for themes after coding. In keeping with a narrative study, the research team performed a retrospective analysis of the journal (Creswell, 2013) by iteratively coding the data. The team started with a combination of categories developed from the previous study (Stewart, Troup, & Plaxco, 2018) and an open coding (Strauss & Corbin, 1998) scheme to allow for the possibility of discovering new categories unique to this study. The main themes for this study were: Teaching, Students, Class Activities, Math (instructor's math, students' math), Reflection, and Tall's worlds. By instructor's math, we mean the math he was doing and talking about, and by students' math, we mean his reflections on students' mathematical abilities and conversations on math in class. For the purpose of this paper, we will only present the analysis from the instructor's journals.

Results

In this section, we will analyze the instructor's journals on five class periods of an introductory linear algebra course, during which the fundamentals of eigentheory were presented. The class met three times each week for a period of 50 minutes. The classes were structured around a sequence of four tasks designed by the Inquiry-Oriented Linear Algebra (IOLA) project (Wawro et al., 2013). The tasks use the ideas of "stretch direction" and "stretch factor" of a linear transformation to develop the formal notions of eigenvector and eigenvalue. Several of the requisite concepts, such as bases, coordinates and matrix representations of linear transformations, were covered earlier in the term so that the IOLA sequence could be used. In analyzing his 5-day teaching segments, we will examine the instructor's (a) movements between Tall's (2013) worlds, (b) pedagogical decision-making moments, and (c) reflections on self and students.

An Analysis of the Teaching Episode: Day 1 (March 30) - IOLA Task 1

The first IOLA task (see figure 1) describes a linear transformation geometrically, in terms of "stretch directions" and "stretch factors," and presents three questions related to it. First, the students are asked to sketch the image of a figure "Z" centered at the origin. In the second part, they are asked to sketch the image of two particular vectors and then compute the precise images. Lastly, they are asked to produce a matrix representation of the linear transformation.

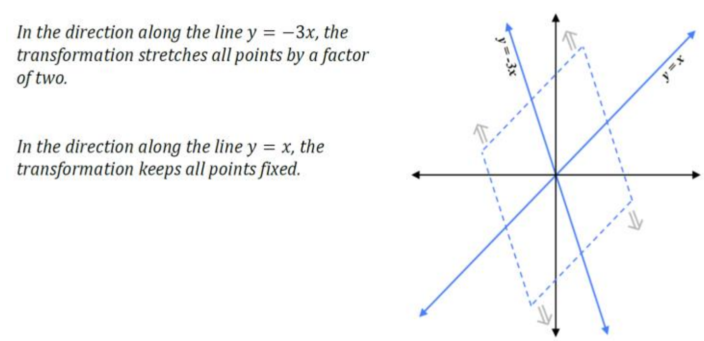

Figure 1. IOLA Task 1

This task is primarily situated in Tall's (2013) embodied and symbolic worlds. By withholding any matrix representation of the transformation, the task was meant to force students to interpret the action of the transformation on vectors via the embodied world. Ideally, this would build intuition and facility. The instructor very quickly noted that students were having difficulty with embodied thinking and decided to take a more active role in guiding them through the task on the board. His next intention was to move students to a more symbolic representation of an idea of stretching, which he wrote as a "mathematical one." The instructor mentioned in his journals that the students struggled again.

> We needed to iron out the common misunderstandings: for every linear transformation the zero vectors get sent to the zero vector, points are identified with vectors, etc. Then we needed to understand what stretching means. After one or two attempts and a geometric description, I asked for a mathematical one. Although no one could articulate it precisely, at least one student had the right idea: scalar multiplication.

In question 2, the instructor computed (symbolically) the images of vectors under the transformation, and had a feeling that students were able to follow. However, their understanding faltered when the instructor changed the vectors slightly. "So, in question 2, we converted the two vectors into linear combinations of vectors in the stretching direction, then used the linearity of the transformation to find their images. I'm not sure if this made sense to them." In question 3, students did not give much feedback. The instructor gave a handout—the preview of the next task— and hoped that "…perhaps the motivated student [would] see the connection of how to use it and then be more prepared for the next task."

An Analysis of the Teaching Episode: Day 2 (April 2) - IOLA Task 2

The second IOLA task continued to build the concept image in much the same way as the first, but instead of a figure "Z", there is a collection of discrete points (see figure 2). Moreover, both the standard coordinate grid (referred to as the "black" coordinates) and the one determined by the eigenvectors (referred to as "blue" coordinates) are overlaid on the collection of points.

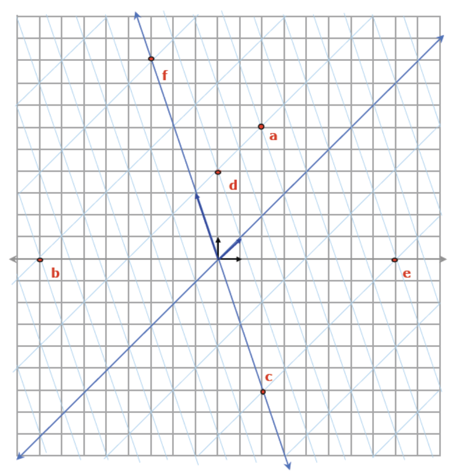

Figure 2. IOLA Task 2.

At the start of the task, the instructor perceived that the students were not engaging with the tasks in a meaningful way. He remarked on having "difficulty getting the students to be active participants." As a result, he "decided to do the worksheet together," meaning that he would guide the class by doing the various parts at the board. He conjectured that "part of the reason that the worksheet took so long was because most students don't have a facility with coordinate vectors."

The instructor made the pedagogical decision before the class started to present the definition of eigenvalue and eigenvector after the first two tasks. Two class periods exploring the connection between coordinates and linear transformations would be sufficient as "a segue to define eigenvalues and eigenvectors." Introducing them halfway through gives some resolution to the first two tasks, while also providing a framework within which the last two tasks can be situated.

Despite recognizing the importance of everyday thought modes for developing concept images, the instructor still views the definition as the most important element in the concept image. Not only does he choose to present it after only two class periods, but he also expresses frustration at not arriving at the definition sooner. “Finally, I was able to define eigenvalue and eigenvector.” In fact, he makes the decision to cut short the discussion of Task 2, Part 3 in order to present the definition. He remarked, “Problem 3 was useful, and I wish I had more time to go through it.”

An Analysis of the Teaching Episode: Day 3 (April 4) Lecture

The instructor made the pedagogical decision to use Day 3 not for the next IOLA task, but instead to synthesize the various embodied, symbolic and formal aspects of eigentheory that the students have so far encountered. To do so, he used exclusively a lecture teaching style. First, he showed how the black and blue coordinate matrix representations of the transformation from those tasks are related by conjugation by the change of coordinate matrix. Next, starting with the standard coordinate representation of the linear transformation, he used GeoGebra to demonstrate visually the effect of the linear transformation on vectors in the unit circle, and in particular how it exactly stretches some, but not all, directions. At this point, he reiterated the eigenvalue and eigenvector definitions and derived the standard way of computing them from the characteristic polynomial and finding the nullspace of $A - \lambda I$. From here, he presented a series of examples including the transformation from the IOLA tasks, an eigenspace with more than one dimension, and the differentiation operator acting on function spaces.

The instructor did not make any remarks on how the students responded to the lecture. Instead, his journal entry was a rather clinical report of the content from the lecture, mainly including the instructor’s math and no mention of students’ math. From this, one could infer that the instructor was engrossed in conveying his own concept image and how he experiences the mathematical concepts of eigenvalues and eigenvectors.

An Analysis of the Teaching Episode: Day 4 (April 6) - IOLA Task 3

On Day 4, the instructor returned to the IOLA sequence with task 3. This task is the most similar to standard textbook exercises for eigentheory. For three distinct two-by-two matrices, the students are asked to 1) find the stretch factors given the stretch directions, 2) find the stretch directions given the stretch factors, and 3) find both the stretch factors and directions. After observing their work for the first part, the instructor noted that, even though “they had a WebWork assignment due the same day that was mostly about computing eigenvalues and eigenvectors,” he “was surprised to see how many were unsure where to start.” The WebWork assignment he mentioned contained only column vectors and matrices, while the IOLA task describes stretch directions. Hence, the instructor interpreted this as a lack of synthesis between the ideas of “direction” and “column vector.” This motivated the instructor’s pedagogical decision to use the blackboard to guide the class through the task, reinforcing certain connections in the image concept.

First, he “decided to go slowly through some fundamental concepts that might be getting in the way of using the eigentheory.” Among the fundamental concepts that the instructor covered were the embodied-symbolic connection between nonzero vectors and “directions” in the plane. Next, he reiterated how shapes in the plane could be thought of as collections of vectors. “I think it’s always worth repeating that a vector ‘lies in a shape or object’ if the tail sits at the origin and tip sits at a point in the shape.” Also, he showed the class how finding the stretch factor (given the stretch direction) is equivalent to solving a linear system with one unknown and usually more than one equation. The fact that the system is consistent is remarkable. With these fundamental notions in place, he proceeded with the work of completing the task. As on Day 3, there was no mention of students’ math in his journals.

An Analysis of the Teaching Episode: Day 5 (April 11) - IOLA Task 4

The fourth IOLA task aimed to introduce students to a subtlety, thus far hidden, of eigentheory: multiplicity. The entire task involved a single linear transformation of R^3, presented as a matrix. As in the previous task, the first two parts involved finding either a stretch direction or a stretch factor, given the other. In particular, it is found that a certain stretch factor has two stretch directions; i.e., the corresponding eigenspace is two-dimensional. The third and final part poses a rather provocative question: given that 2 and 3 are stretch factors and the former has two distinct stretch directions, could there be additional stretch factors? At the heart of this question is the observation that eigenvectors for distinct eigenvalues must be linearly independent. A counting argument then shows that we already have a basis of eigenvectors and hence there can be no other eigenvalues.

The instructor appeared eager for the class to spend time with this last part. He made the pedagogical decision to go through [the first two parts] together on the board. “My hope was that this would put everyone on the same page to try the third part.” Once the students had an opportunity to think about the third part, he observed:

> Every student’s work that I saw was the same. To decide if there was another eigenvalue or stretch direction they all computed the characteristic polynomial to see if there was another root. I anticipated this, so I then presented a solution that crucially uses the fact that all three eigenvectors form a basis for R^3. I did not get very much feedback from the class on whether they were internalizing this.

Although there were multiple ways to approach the third part, the students all reached for the most symbolic, computable solution. They found the characteristic polynomial in order to find all the eigenvalues; anticipating this, he presented a contrasting formal solution. In this way, the students would hopefully see alternatives to the symbolic world, and perhaps build a connection between the two concepts of basis and eigentheory. The final piece of eigentheory was diagonalization. After presenting an example with insufficiently many stretch directions, the instructor was in a position to explain diagonalization and when it can be done.

Discussion and Concluding Remarks

Throughout the course, the instructor tried to follow IOLA’s objectives designed for each task. His intention was to have the students work in small groups to complete each task first, and then come together as a class to discuss solutions. However, on many occasions when he noticed that progress among the students was much slower than anticipated, he often reverted to a more standard lecture format. While encouraging participation from the class, he would go through the tasks at the blackboard.

While the instructor’s decision to use the IOLA tasks shows that he values the embodied and symbolic worlds as part of the concept image, the instructor’s goal of reaching the formal world became apparent in many of his journal writings. For example, on Day 5, the instructor tries to speed through what he considers “rote” so that the class can get to something more formal that generates connections between concepts. In fact, his decision to present the definitions of eigenvalue and eigenvector at precisely the midpoint of the unit reflects the significance they hold for him. They represent, for the instructor, a single idea which unites the various notions from all three world that the students have been exposed to. A mathematical understanding of eigentheory (to him) involved primarily the definitions, but also how those definitions manifested themselves in the embodied and symbolic worlds. The instructor’s objective was a mathematical treatment of eigentheory, so he used IOLA to present a web of connections surrounding the formal definitions.

The instructor seemed to believe the more connections between eigentheory and other linear algebraic concepts that he could convey to the students, the more robust the concept image. He laments not showing how the linear system that must be solved to obtain the stretch factor would be inconsistent if it was set up with a non-stretch direction. "What I should have done, in addition, is to point out that when you choose a vector, not in one of the eigenspaces, then solving for a stretch factor will lead to an inconsistent system." Later, he regrets not connecting the formalism of solving linear systems to finding eigenvectors. "But perhaps I should have gone through the derivation of the nullspace of a matrix, rather than appealing to their experience with WebWork calculations." It was interesting to notice that even a mathematician that values all three worlds of mathematical thinking would still gravitate more toward the formal world as the most important part of a mathematical concept. Although he recognizes the necessity of embodied and symbolic concepts in the acquisition and understanding of mathematical concepts, they are merely the scaffold on which the formal notions are built.

In Tall's view, "formal mathematics is more powerful than the mathematics of embodiment and symbolism, which are constrained by the context in which the mathematics is used" (2013, p. 18). Due to this added power, Tall believes that the formal world can interact with and inform the embodied and symbolic worlds. In particular, in his view, "formal mathematics can reveal new embodied and symbolic ways of interpreting mathematics" (p. 18).

Lastly, one may speculate that the instructor underestimated the time necessary for establishing new connections between mathematical ideas. What appears "rote" and part of his "everyday" mode of thinking is completely foreign to the typical undergraduate linear algebra student. Hence, the connections between the formal definitions and surrounding concepts that appeared so strong to the instructor were quite tenuous with the students. Perhaps this accounts for the frustration that appears in his tone and the decision in multiple instances to explicitly guide the students to the connections.

The research team is in the process of analyzing the data from students' surveys as well as analyzing the instructor's journals on more linear algebra concepts. It will be interesting to know which connections, if any, were effective in developing the students' concept image. Do the connections between and within Tall's worlds benefit the teaching of eigentheory?

References

Caglayan, G. (2015). Making sense of eigenvalue–eigenvector relationships: Math majors' linear algebra – Geometry connections in a dynamic environment. *Journal of Mathematical Behavior* 40, 131–153.

Creswell, J. W. (2013). *Qualitative Inquiry and research design: choosing among five approaches*, (3rd ed). SAGE.

Davis, E. (2006). Characterizing productive reflection among preservice elementary teachers: Seeing what matters. *Teaching and Teacher Education* 22, pp. 281–301.

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educative process*. Boston, MA: D. C. Heath & Company.

Fund, Z. (2010). Effects of communities of reflecting peers on student-teacher development – including in-depth case studies. *Teachers and Teaching: Theory and Practice,* 16, 679-701.

Gol Tabaghi, S., & Sinclair, N. (2013). Using dynamic geometry software to explore eigenvectors: the emergence of dynamic-synthetic-geometric thinking. *Technology, Knowledge and Learning*, *18*(3), 149–164.

Moore-Russo, D., & Wilsey, J. (2014). Delving into the meaning of productive reflection: A study of future teachers' reflections on representations of teaching, *Teaching and Teacher Education* 37, 76-90.

Salgado, H. & Trigueros, M. (2015). Teaching eigenvalues and eigenvectors using models and APOS Theory, *The Journal of Mathematical Behavior*, 39, 100-120.

Stewart, S. (2018). Moving between the embodied, symbolic and formal worlds of mathematical thinking in linear algebra. In S. Stewart, C. Andrews-Larson, A. Berman, & M. Zandieh (Eds.) *Challenges and Strategies in Teaching Linear Algebra*. (pp. 51-67). Springer International Publishing.

Stewart, S., Troup, J., & Plaxco, D. (2018). Teaching Linear Algebra: Modeling one instructor's decisions to move between the worlds of mathematical thinking. In Weinberg, A., Rasmussen, C., Rabin, J., Wawro, M., & Brown, S. (Eds.) *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education*. (pp. 1014-1022). San Diego, CA.

Stewart, S. Thompson, C. & Brady, N. (2017). Navigating through the mathematical world: Uncovering a Geometer's thought processes through his handouts and teaching journals. In Dooley, T. & Gueudet, G. (Eds.) *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education*, (pp. 2258-2265) CERME10, February 1 – 5. Dublin, Ireland: DCU Institute of Education and ERME.

Strauss, A. L., & Corbin, J. (1998). *Basics of qualitative research: Grounded theory procedures and techniques* (2nd ed.). Newbury Park, CA: Sage.

Tall, D. O., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12(2), 151-169.

Tall, D. O. (2010). Perceptions Operations and Proof in Undergraduate Mathematics, Community for Undergraduate Learning in the Mathematical Sciences (CULMS) Newsletter, 2, 21-28.

Tall, D. O. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics*, Cambridge University Press.

Thomas, M. O. J., & Stewart, S. (2011). Eigenvalues and eigenvectors: Embodied, symbolic and formal thinking. *Mathematics Education Research Journal*, 23 (3), 275-296.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. O. Tall (Ed.), Advanced mathematical thinking (pp. 65-81). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Wawro, M., Zandieh, M., Rasmussen, C., & Andrews-Larson, C. (2013). Inquiry oriented linear algebra: Course materials. Available at http://iola.math.vt.edu. This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.