Analyzing the Nature of University Students' Difficulties with Algebra in Calculus: Students' Voices during Problem Solving

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The aim of this research was to investigate the nature of difficulties with algebra in calculus problems from the perspective of students. We employed Skemp's (1979) theory to analyze two calculus students' difficulties with algebra in an interview setting. Our findings indicate that although these students were aware of their challenges with algebra, they struggled to resolve those issues in the context of calculus. Likewise, both seem to struggle in different ways with algebra outside the context of calculus. Implications for teaching based on our current research will be provided.

Keywords: algebra, calculus, path, director system, schema

Theoretical Background

Although, research on students' difficulties with school algebra has been prolific (e.g. Ashlock, 2010; Booth, Barbieri, Eyer, & Pare-Blagoev, 2014; Kieran, 1992; Hoch & Dreyfus, 2004; Stacey, Chick, & Kendal 2004), and students' difficulties with Calculus (e.g. Bressoud, Mesa, & Rasmussen, 2015, Tallman, et. al, 2015) has been conducted, research on students' school algebra shortcomings in calculus courses are scarce. Reeder (2017) addresses the fact that while students may be successful with algebra in high school, they often leave high school with a shallow, inflexible understandings. While understanding why and how this gap in student mathematical knowledge and skills exists is a complex endeavor, the fact that it does exist, is commonly known. Universities are keenly aware of the mathematical challenges of students entering the university. In light of this, many interventions have been developed to help students fill the necessary gaps in the mathematics knowledge and skills needed to be successful in university level mathematics courses. Unfortunately, according to McGowen (2017), one of the most common interventions, remedial mathematics courses, is not providing the needed support students need to be successful in university mathematics courses.

Recent research by the authors sought to understand the nature of student challenges with algebra in calculus settings. In a study by Stewart, Reeder, Raymond, & Troup (2018), participants in a Calculus I course were asked to solve a set of calculus questions and corresponding algebra questions that paralleled the algebra needed in the calculus questions. The findings of this study revealed, many students struggled with the algebra, inside and outside of the calculus context. Our research shows that many students, when confronted with algebra in a calculus context, tried to avoid the algebra required to solve the problem while others attempted the algebra and lost their way resulting in their inability to complete the problem correctly.

To examine the nature of the algebra difficulties in calculus context, for this study, we will employ Skemp's (1979) model of intelligence presented in his book, *Intelligence, Learning, and Action*. In remembering his work, most readers will recall his *relational understanding* and *instrumental understanding* (Skemp, 1976). Later, Skemp (1979) devoted an entire chapter on understanding (Chapter 10) as he developed his idea of a *schema*.

In conveying his theoretical ideas and connecting to his audience, Skemp (1979) made use of many everyday examples. In our theoretical stance we will draw on a coherent segment of his model and utilize some of his examples applicable to this study, in order to analyze calculus students' mathematical thinking and actions. Skemp's model claimed that most human activities are for survival and therefore goal orientated. In order to explain how humans organize their actions, he used the metaphor of a *director system,* which is central to his model. He defined a director system "that which directs the way in which the energy of the operator system is applied to the operand so as to take it to the required state and keep it there. .. for the rudder it is a valve mechanism" (p. 41-42). By an operand he meant, "that which is changed from one state to another and kept there…e.g. a ship's rudder, which is brought to the desired position and kept there" (p. 41). He defined "operators, as that which actually does the work of changing the state of the operand (..the position of the rudder) from its initial state to the state chosen by the …helmsman.)" (p. 41). In Skemp's view:

> Using swimming as an example, a non-swimmer is outside his prohabital if he is in deep water, not because of lack of muscular strength but because he cannot make the right movement. He is within the capacity of his operators but outside the domain of his (relevant) director system. A good swimmer caught in an off-shore current is also outside his prohabitat but for a different reason. He can make the right movements, but cannot swim powerfully enough to reach the shore, or he cannot keep it up for long enough. So he is within the domain of his director system, but outside the capacity of his operators. Both are non-viable because they are outside their prohabitats; but for different reasons. (p. 62).

Skemp defined the prohabitat as "… that region which is within both the domain of the director system and the capacity of the operators" (p. 62). Within Skemp's model he defined the idea of *knowing that*, as possessing an appropriate *schema*. In his views a "schema is a highly abstract concept" (p. 167). He defined "a *path* as a sequence of states and a *plan* consists of (i) a path from a present state to a goal state; (ii) a way of applying the energies available to the operators in such a way as to take the operand along the path" (p. 168). He further described "the connection between *knowing how* and *being able* to is the connection between having arrived at a plan, and putting it into action" (p. 184). In his view, "prerequisite for the production of these plans is understanding: the realization of present state and goal state within an appropriate existing schema" (p. 170). Some researchers have employed Skemp's model and drawn from his work. For example, Olive and Steff (2002, p. 106) used Skemp's work to build "a theoretical model of children's constructive activity in the context of learning about fractions." Berger and Stewart (2018) employed his idea of *schema*, to describe students' proofs in an introductory topology course.

The purpose of this study is to investigate student thinking as they encounter algebraic problems within a calculus context in order to shed light on the origin of these difficulties. More specifically, this study sought to answer the following questions: How did students react/respond to algebra processes in the context of calculus problems? What were their plans and what paths did they take to reach their goal state?

Method

The study employed a qualitative case study methodology for data collection and analysis using Skemp's (1979) model as a framework for making sense of the data. Students enrolled in Calculus I at a large university in the South Midwest United States were invited to participate in an one-on-one interview wherein they would be asked to complete a few problems and discuss their strategies and challenges with those problems (see Figure 1). Students were recruited from multiple Calculus I classes early in the semester. If interested in participating, they were asked to provide their name and email address. A member of the research team contacted each student and arranged for a time for a two-hour interview. Ultimately, four students participated in these interviews. Each participant student was given three common Calculus I tasks and were asked to choose two to complete. Based on the participants choices of calculus tasks, they were then given an additional two algebra tasks which mimicked the algebra skills needed in the calculus tasks. As they solved each of these four tasks, they were asked to think-aloud and describe what they were doing or thinking about. After students completed the four tasks, a semi-structured interview was conducted to further investigate students' thoughts, perceptions, confusions, and frustrations. The questions posed in the interview sought to elicit more of the participants' perceptions and thinking. Probing questions were asked by the interviewers when warranted by participants' responses. Once all data were collected, it was de-identified and think aloud interviews were transcribed verbatim. The data were analyzed using a variety of themes drawn from Skemp's model as described in this paper.

1. Evaluate $\lim_{x\to 3} \frac{x^2-9}{\sqrt{x}-3}$ 2. Use the limit definition of the derivative to find $f'(x)$ for $f(x) = (x-4)^2 - x$ 3. Find the critical points of $f(x) = \frac{-(x^2+1)}{3x^2-6}$. 4. Simplify $\frac{x^2-9}{\sqrt{x}-9}$ by rationalizing the denominator. 5. Find $f(3+x)$ if $f(x) = \frac{(x-4)^2-x+4}{x-3}$ simplify your results by eliminating parentheses and collecting like terms. 6. Use the distributive property to solve $\frac{-2x(3x^2-6)-6x(-(x^2+1))}{(3x^2-6)^2} = 0$.	1. What are your thoughts about the problems you just completed? 2. Which problems were particularly challenging? 3. How do you think you did on the problems? 4. Would you like to ask any questions about these problems? 5. What, if anything, do you find challenging about MATH 1823? 6. Do you feel you were well prepared for MATH 1823? Why or why not? 7. How confident do you feel in your ability to complete calculus problems? 8. How confident do you feel in your ability to complete algebraic problems?

Figure 1. Tasks and questions for student interviews.

Results

Based on our prior research (Stewart & Reeder, 2017; Stewart, et al, 2018), we have established two common types of calculus problems with corresponding algebra occurrences in those problems. These are presented as problems wherein the calculus proceeds the algebra (Type 1) (see figure 2 #1) and wherein the algebra proceeds the calculus (Type 2) (see figure 2 #2). Analysis of both these common types of problems presented in Calculus I classes, reveals that in Type 1 calculus problems, many students can take the first derivative, but are not able to carry out the many steps of algebra to complete the problem (Stewart & Reeder, 2017). Likewise, analysis of Type 2 calculus problems, reveals that many students either try to avoid the algebra in the first steps altogether, or have difficulty with the algebra that often involves rationalizing the denominator, factoring, which results in incorrect answers (Stewart, et al, 2018).

Calculus: Remember: • Use the product rule • Use the chain rule Algebra: Simplify the derivative equation by adding the two algebraic expressions we find via common denominators. Calculus: To find the critical numbers, set $f'(x) = 0$ Algebra: Factor to find roots of the quadratic polynomial. Calculus: Understand that to find the absolute max and min, we have to test the critical points and the boundaries. Algebra: Actually test the boundaries and critical points by evaluating the function at the appropriate values.	**1. Find the absolute maximum and absolute minimum values of $f(x) = x\sqrt{x - x^2}$** Differentiating yields: $= x\dfrac{1-2x}{2\sqrt{x-x^2}} + \sqrt{x-x^2}$ Combine into a single fraction $= \dfrac{(x-2x^2)+(2x-2x^2)}{2\sqrt{x-x^2}} = \dfrac{3x-4x^3}{2\sqrt{x-x^2}}$ Set $f'(x) = 0$ $\dfrac{3x-4x^3}{2\sqrt{x-x^2}} = 0$ $\Rightarrow 3x - 4x^2 = 0 \Rightarrow x(3-4x) \Rightarrow x = 0 \text{ or } x = \frac{4}{3}$ Test boundaries and critical points $f(0) = f(1) = 0$ absolute minimum $f\left(\dfrac{3}{4}\right) = \dfrac{3}{4}\sqrt{\dfrac{3}{4} - \left(\dfrac{3}{4}\right)^2} = \dfrac{3}{4}\sqrt{\dfrac{3}{16}} = \dfrac{3\sqrt{3}}{16}$
Calculus: Recognize the denominator prevents us from evaluating the limit directly. Algebra: Recognize the common factors in numerator and denominator, rationalize, factor, and cancel these common factors. Calculus: Recognize that since the function is continuous, the limit can be evaluated directly. Algebra: Evaluate the limit directly by setting $x = 3$ and computing the result.	**2. Evaluate $\lim\limits_{x\to 3} \frac{x^2-9}{\sqrt{x-3}}$** Rationalize the denominator: $\lim\limits_{x\to 3} \dfrac{x^2-9}{\sqrt{x-3}} * \dfrac{\sqrt{x-3}}{\sqrt{x-3}} = \lim\limits_{x\to 3} \dfrac{(x^2-9)\sqrt{x-3}}{x-3}$ Factor the numerator: $\lim\limits_{x\to 3} \dfrac{(x+3)(x-3)\sqrt{x-3}}{x-3}$ Cancel the common factor: $\lim\limits_{x\to 3} \dfrac{(x+3)\cancel{(x-3)}\sqrt{x-3}}{\cancel{x-3}} =$ $\lim\limits_{x\to 3} (x+3)\sqrt{x-3}$ Evaluate at $x = 3$ $= (3+3)\sqrt{3-3} = 6(0) = 0$

Figure 2. Type 2 Calculus Problem (#1): Taking the first derivative (calculus) followed with many steps of algebra. Type 2 Calculus Problem (#2): Many algebra steps followed by the final step of taking the limit (calculus).

While participants in this study were given both Type 1 and Type 2 problems, for the purpose of this paper, we will only focus on one Type 2 calculus problem that all participants selected to solve. We will share findings resultant from the interview questions with two participants. The problem that all participants selected to solve was a limit problem requiring students to begin by rationalizing the denominator in order to get started (see Figure 2). The results will be presented as two cases.

Student 1 Case. Thinking aloud, Student 1 wrestled with solving this problem. He shared:

… finding the limit as it pushes 3 but I can't put 3 in right now because that will put zero in the denominator and that doesn't work so I have to do something to this to make it work. So, I'll multiply both the top and the bottom by the square root of x minus 3... yeah... that way I can get rid of the square root in the problem? Yeah. And then I'll be able to work with the x minus 3 in the bottom. So, then to just get rid of the bottom part of the fraction... yeah... I can... yeah multiply the top and bottom by x minus 3 because that's the same as multiplying by 1... I think... yeah it is... it is. And then, I still have the square root of x minus 3 on the top, and there's nothing on the bottom, there's 1... or... no that doesn't work... because then I just have a different factor on the bottom... I'll just have that squared.

When challenges with algebra were encountered, he began to re-think his process:

"I'll just start over. I don't think that first stuff was right anyway. There's probably something I could do with the conjugate but I don't remember, I don't know if that applies here. I don't think it does. Maybe it does. Um... I can... I can factor the top that's what I can do. Actually no, I'll go back I'll do the same first step again... that works... and so I'll do... multiply both the top and bottom by the square root of x minus 3, so that gets rid of the square root on the bottom and it's just x minus 3 and then I can factor... yeah factor the x^2 minus 9 into... because it's the whatever the difference of squares or something... it just works out... and that way I can cancel out the x minus 3 on the bottom now and I can take the limit with what I just have... that I can insert 3 into. So now I just have the x plus 3 times the square root of x minus 3. And I just plug in 3 because this is... this is real everywhere I'm pretty sure...yeah... yeah... no I can't. Can I? I don't think I can. Because I still have a problem with the square root on the top now. Maybe. Hmmm... no I can, I can take the square root of 0 that's fine. That's just 0. So... it's 3 plus 3, the square root of 3 minus 3 which is 0, so that's 6 times 0 which is just 0. So that's limit. Yeah."

This student made a plan and took a path that was not helpful, in trying to solve the limit problem. He then revised his plan and took a different path and was able to find the correct answer. In analyzing his work, we noted the connection between Skemps's *knowing how* and *being able to*. Although, he had a plan, due to the lack of algebra knowledge available, he was not able to reach the goal state on that chosen path in his first attempts at solving the problem.

When this student was asked about how they felt about the problems he had just solved, he immediately noted his challenges with algebra while working on calculus. "I don't know I think I just struggle with problems like this because it's hard for me to see what to do. Which I don't really know why because it's just... like... algebra. I don't know." This again can be made sense of in terms of an inability to determine a *path* when confronted with algebra in the context of calculus. When asked about which problems he felt more confident with, the calculus or the algebra, he indicated the calculus. This is interesting given the fact that the algebra problems paralleled the algebra needed with in the calculus problems demonstrating again Skemp's *knowing how* but not *being able to*. Clearly the student knows how to complete the algebra and has done so successfully many times but is not able to for these problems.

Student 2 Case. Student 2 initially approached the limit problem by trying to evaluate the limit without completing any algebra. After a few minutes, however, he began to try to simplify the problem. He shares his thoughts as he attempts to solve the problem and notes that he cannot recall how to complete this problem because it has been a few weeks since limit problems were the focus of study in class:

> …for evaluating the limit of x goes to 3, function being x squared minus 9 over root x minus 3. So, to solve this one... um... I am going to... let's see... I guess I could divide by the highest power of x which in the denominator is root x squared... yeah... hang on... do I need to do that? Well, … for the limit to exist the left-hand limit has to equal the right hand limit so... um... as x is approaching 3 from the negative side, from the left um... our denominator is getting closer and closer to 0. But that's going to be slightly less, so it's um... approaching 0 from the... the left-hand side though. But I don't think... that is only going to tell me if there's asymptotes in the graph, if I recall. Um... I mean it's been a while since I've done limits. And if we divided by the... I'm just going to go ahead and divide everything by the highest power of x but... yeah no... I don't want to do that. Um... Yeah, I would have to... honestly, I really don't remember... and I would have to... I would need to jog my memory... Which I think, I mean I've done them... if I were to jog my memory I think I would... I don't think I would have too much of an issue but... everything from the beginning of the semester I have really put on the backburner and I need to bring it back.

Avoiding algebra is one of the cases we see often in calculus questions (Type 2). This student made a plan to “divide by the highest power of x”. He then questions that plan and abandoned it. Then he recalled some limit laws, and at this stage he is not thinking about performing any algebra, rather thinking more formally. Failing that, he decides to go back to his original plan and “divide everything by the highest power of x”. However, his lack of algebra again lets him down. He does not say, how should I do that, instead he says: “I really don’t remember”, hence, he is not able to action his plan. Unfortunately, this student was not able to reach his goal state of solving this problem.

When this student was asked about how he general felt about the problems he indicated that he had some difficulty with the limit problem given it had been a few weeks since he had worked on them in class. “I think it’s just that I haven’t …done these honestly, not .. it hasn’t been that long … a couple of weeks? And the thing is … I know how to do them, but I do not know it well enough…” In this case, utilizing Skemp’s swimming example, we can see that the student knows how to swim but while swimming in new waters, or having not gone swimming for some time, he is unable to swim well.

Discussion and Implications

Calculus courses are widely considered a gateway to disciplines in Science, Technology, Engineering, and Mathematics (STEM), and as such, have garnered particular attention. Negative experiences encountered in gatekeeper or introductory math and science courses are significant contributors to more than half the attrition of declared STEM majors (Crisp, Nora, & Taggart, 2009; Mervis, 2010). In this way, calculus course often act as a significant obstacle or one that discourages students from pursuing STEM majors (Bressoud, Mesa, & Rasmussen, 2015).

Prior research (Stewart, et al, 2018) revealed that students' algebraic challenges included problems working across the balance point in equations, cancelling, operating with radicals, distributing, and incomplete algebra. That incomplete algebra was one of the most common errors was also both interesting and puzzling. Despite knowing what types of mistakes students were making it was difficult to ascertain *why* they made the mistakes. This study sought to better understand why calculus students make mistakes with algebra.

Utilizing Skemp's (1979) model to make sense of students' work helped to frame a better understanding of why calculus students are challenged with algebra. Skemp's swimming example is particularly useful. Successful students are also those strong swimmers who are able to swim within their boundary. They can swim regardless of the water, whether it is deep and unfamiliar, or shallow and calm. These students can work successfully with algebra within or outside a calculus context. They know what to do and are able to do it. Likewise, successful students are able to recognize when they are not being successful and choose a different path. According to Skemp (1979), "the greatest adaptability of behavior is made possible by the position of an appropriate schema, from which a great variety of paths can be derived." (p. 169). Unfortunately, the majority of students are not strong swimmers. Despite having completed several years of high school algebra and being placed in a university calculus course, many students seemingly *know how to* but are *not able to* successfully deal with the necessary algebra needed for most calculus problems. The students in this study would often begin a path but it would not lead them to their goal state.

In dealing with limit problems specifically, most instructors agree that first year calculus students struggle with conceptual and procedural aspects of limits. However, the nature of these struggles are not known. We believe that theorizing the situation will give insight in understanding the extent of students' difficulties and interventions for instruction. We also believe that more research in understanding students' difficulties with algebra in calculus is needed.

We agree with Tall (2017, p. 61) who suggests that mathematicians, curriculum designers, teachers, and learners need "to become explicitly aware of the underlying supportive and problematic aspects of long-term learning". Reeder (2017) suggests college instructors face the challenge of working with students everyday who can seemingly make sense of complex mathematical concepts but are unable to solve problems related to those concepts due to their difficulties with algebraic procedures. While resolving the algebra deficiencies that students bring with them will be challenging, "it cannot be simply ignored and remain as an everyday accepted or out of our hands part of teaching university level mathematics courses" (Reeder, 2017, p. 15).

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