The Relational Meaning of the Equals Sign: a Philosophical Perspective

Alison Mirin
Arizona State University

While there has been research on students' understanding of the meaning of the equals sign, there has yet to be a thorough discussion in math education on a strong meaning of the equals sign. This paper discusses the philosophical and logical literature on the identity relation and reviews the math education research community's attempt to characterize a productive meaning for the equals sign.

Keywords: Equals Sign, Equality, Identity, Equation

The study of students' understanding of the equals sign has been a long-standing theme in mathematics education research. This line of inquiry predates mathematics education's existence as a well-formed field, and the philosophical literature addressing the meaning of the equals sign dates to the late 19th century or earlier (Renwick, 1932; Noonan & Curtis, 2014). There is extensive discussion on the meaning of the identity relation that predates the equals sign and continues to this day. This discussion appears as early as the 4th century BCE in Plato's *Parmenides* and as late as Williamson in the 20th century (Noonan & Curtis, 2014). In spite of the long history of the research and development of ideas associated with identity, math education literature is plagued with vagueness and carelessness in characterizing the meaning of the equals sign. Despite this lack of rigor, the field of math education has produced compelling research revealing that students have weak understandings of the equals sign (Baroody & Ginsburg, 1982; Behr, Erlwanger, & Nichols, 1980; Byrd, McNeil, Chesney, & Matthews, 2015; Denmark, Barco, & Voran, 1976; Kieran, 1981; Oksuz, 2007; Sáenz-Ludlow & Walgamuth, 1998).

The Importance of the Identity Relation

Identity statements, and their assessments of sameness, are an important part of mathematics. This is especially evident when we consider the prevalence of the equals sign, and there is a body of math education literature addressing students' understanding of it. This paper serves as a survey and critique of that existing literature, together with a philosophical discussion on the meaning of the equals sign. The importance of the equals sign in mathematics cannot be understated. There is not one branch of mathematics that does not rely on it. Consider, for example, the equation "2+2=4" or "$cos(x) = \Sigma^{\infty}{}_{0} \quad (-1)^n x^{2n}/(2n)!$".

One way to view identity statements is to see them as giving multiple representations of the same thing. "The evening star" is a representation of the planet Venus (in the sense that it refers to Venus), as is "the morning star", and the statement of identity tells us that in fact these phrases refer to the *same* thing, despite being *different* representations.

Having multiple representations of the same thing plays an important role in mathematics – for example, in combinatorial arguments where we find two ways to calculate the same thing, resulting in an identity statement. A quick examination reveals that represents the number of ways to choose k items from a group of n items, and represents the number of ways to leave out n-k items from a group of n items. Hence, we see that =, which is indeed an informative and useful statement.

When we deal with mathematical statements, we run into another conundrum: what *thing* does the name of a function or number refer to? Clearly, the name of a function refers to a function, and the name of a number refers to a number, but what is a number, and what is a function? Numbers are not physical objects *out there* that we can easily point to. When we say that 7+2=8+1, what is it that we are saying is the same? That is, what are the referents of each side of the equation? Mathematicians and philosophers cannot agree on the existence of numbers as objects, let alone what numbers *are* (Horsten, 2016). Yet, despite there being an age-old question of what an abstract mathematical object *is* (if anything), mathematicians continue to reason productively using equations, identity statements, numbers, and functions.

Consider the notion of a set; in Zermelo-Frankel (ZF) set theory, which is arguably the basis of much of mathematics, the word "set" is undefined (Bagaria, 2016). We may *think* of it as a collection of objects. However, we infer *sameness* of sets via the axiom of extensionality, which says that two sets are identical if and only if they have the same elements. This requirement of sameness is what allows mathematicians to identify the set {3,2} as the same as the set {2,3}, By coming up with this criterion of sameness, mathematicians can define the set {3, 2} as the same as the set {2,3} while remaining agnostic about what a set *is* ontologically. We can see that criteria for sameness can give math inferential power.

The Meaning of the Equals Sign

For mathematicians, the equals sign means *is the same as* or *is identical to.* We must be clear about how we use the word "is". There are at least two distinct ways that we use the word "is". The word "is" can refer to identity (i.e. mean the same as "="), but it can also be used as a means of predication ("To Be," n.d.) [1] For example, in the sentence "the morning star is the evening star," the word "is" denotes the identity relation, since the object "the morning star" refers to is the same object as "the evening star". Contrast this with the sentence "Socrates is mortal," in which "is" refers to a property of Socrates. In this paper, I use the word "is" to refer to predication, and for identity, I use "equals", "is the same as", and "is identical to" for emphasis.

There is a standard modern criterion for truth of identity statements: "a=b" is true if and only if the object named by "a" is the same object as the object named by "b" (Frege, 1879/1967; Mendelson, 2009; Noonan & Curtis, 2014). For example, the "president of the United States inaugurated in 2017=Donald Trump" is true because the phrase "the president of the United states inaugurated in 2017" names the same object as "Donald Trump". Similarly, "2+3=4+1" is true because the object named by "2+3" is the same object (the same number) as 4+1 (provided that numbers are objects that exist). This is the criterion that Frege held for truth of equality sentences (Zalta, 2016).

But what do identity statements *mean*, and why are they informative? What does "as the same as" mean? Gottlob Frege addresses these questions using his infamous puzzles in *On Concept and Object* and *On Sense and Reference* (Frege, 1892/1948). One of his puzzles discusses the meaningfulness of the sentence

(1) "The morning star is the evening star".

If nouns mean no more than their referents, then since "the morning star" and "the evening star" both refer to a physical object (the planet Venus), (1) just means

(2) "Venus is Venus".

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[1] Frege wrote a letter to Wittgenstein that his opening line to the Tractatus, "The world is everything which is the case" ("Die Welt ist alles, was der Fall ist") is ambiguous due to not specifying whether the first use of the word "is" ("ist") is used as predication or as identity.

However, (1) is clearly an informative statement, whereas (2) is not. In other words, if noun phrases do nothing more than refer to objects (in this case, the planet Venus), statements of identity are not informative.

Although this is the accepted view, Frege himself struggled with the nature of the equality relation (the "is" of identity). While he never doubted the criterion for truth of "a=b", as described above, he initially considered two possibilities for the *meaning* and nature of the equality relation: (a) that the equals sign expresses a relation between names and (b) that the equals sign expresses a relation between objects. He decided that the equality relation is a relation between names, with the rationale that some identity statements (e.g. "The Morning Star=The Evening Star", 2+2=4) are indeed informative (Dejnozka, 1981; Frege, 1879/1967; Makin, 2010). That is, he decides that "The morning star=the evening star" just *means* that the object that the name "the morning star" refers to is the same object that the name "the evening star" refers to, and that "2+3=4+1" means that "2+3" and "4+1" are names for the same number. However, he later rejects his assertion that the equality relation is a relation between names, on the grounds that the meaning of identity statements would then be statements about arbitrary linguistic convention, rather than expressing what he calls "objective knowledge" (Frege, 1879/1967; Makin, 2010). As a result, he creates a notion of s*ense* as an aspect of a name's meaning (in addition to its *referent*). A name *expresses* a sense and *refers to* or *denotes* its referent. He characterizes a name's sense as a "mode of presentation" of its referent (Makin, 2010). Roughly, a name's sense is what *picks out* its referent. Frege also called sense "cognitive value". A sense is something that we *grasp*: "We relate to a sense by *grasping* it, which is what understanding the attached name consists in" (Makin, 2010). The terms "The morning star" and "the evening star" express different senses but denote the same referent (the planet Venus), and the sentence "The morning star = the evening star" means that the *senses* expressed by "the morning star" and "the evening star" pick out the same referent (the planet Venus). Similarly, "2+3" and "4+1" express different *senses* and denote the same referent, and the sentence "2+3=4+1" means that the *senses* expressed by "2+3" and "4+1" pick out the same referent (the number 5, which is the same as the number 2+3 and 4+1).

To emphasize what I mentioned earlier, equality represents true identity, not merely an equivalence relation: a=b if and only if a *is the same thing* as b. It does not suffice for a to be *equivalent* or *isomorphic to* b. Taking another example from set theory, it is not the case that Z/2Z=Z2. Mathematicians might casually refer to them as "the same group," but they are actually different groups (members of Z/2Z are *sets* of integers, whereas members of Z2 are integers). Z/2Z and Z2 are of the same *isomorphism class,* but they are not *equal* to each other. This is not to say that that are unequal simply because we write members of Z/2Z one way and members of Z2 another way; indeed we can have two different names for the same thing. For example, we can write *the same* group with additive or multiplicative notation, we have the same group, not merely isomorphic. Similarly, we can call the same function both "f" and "g". So long as the set for the group, together with its operation, are identical (despite different names), then the groups are identical.

Hodges (1997), a model theorist, attempts to make a similar point that I am making here: "A group theorist will happily write the same abelian group multiplicatively or additively, whichever is more convenient for the matter in hand. Not so much for the model theorist: for him or her the group with '*' is one structure and the group with "+" is a different structure." (p.1). The main point Hodges makes is that isomorphism and identity ought not be conflated. However, he conflates the notion of isomorphic structures with structures of the same name. By reducing the notion of isomorphism to the notion of naming, there's an important nuance he's missing: that we can have isomorphic, non-identical structures, yet *also* have a structure and name it two different ways. Let's return to the example of groups with two elements: the groups Z/2Z and Z2 are isomorphic but *not* identical, for the reasons described above. Yet, if we wanted, we could write Z2 the typical way, as the set {0,1} together with the function {(0,0,0),(0,1,1),(1,1,0),(1,0,1)},or we could simply say "let a=0 and a=1", and write Z2 as the set {a,b} together with the function {(a,a,a),(a,b,b),(b,b,a),(b,a,b)} and indeed these would be the *same group* while still remaining distinct from Z/2Z.

This may seem like mindless pedantry, however, there are good reasons to be careful about equivalence versus equality. If we conflated equivalence and equality, Galois theory (which involves counting isomorphisms) would be a lot less interesting (Burgess, 2015; Hodges, 1997). Despite the distinction, equivalency and identity do have some properties in common - namely, identity (equality) is a particular equivalence relation in the sense it is reflective, transitive, and symmetric.

I choose to precede the literature review on the equals sign with a discussion of the philosophy of identity to emphasize that appraising the meaning of identity statements is a non-trivial activity. Frege himself considered at least *three* different meanings for the equals sign (relation between objects, relation between signs, relation between senses). It is thus entirely possible that students can also have multiple, nuanced meanings for equations. We can adapt the meanings that philosophers and mathematicians have considered, since they are ways of thinking of the equals sign that may pertain to students' particular mental models

Here is summary of the major philosophical and mathematical points the reader should keep in mind when reading the literature review on the equals sign. The sentence "a=b" is true if and only if the object named by "a" is the same as the object named by "b". While this is a criterion for *truth* of statements of the form "a=b," it does not account for the *meaning* of "a=b," as can be seen by Frege's work. As Frege shows us, discerning the meaning of such statements is nontrivial, and even an esteemed philosopher and mathematician such as himself considered more than one possible meaning. Another important thing to keep in mind is that in mathematics, "=" expresses the relation of identity: that is, "is equal to," "is the same as," and "=" all refer to the same relation. This relation is a specific example of an equivalence relation, but there are other equivalence relations that are not this specific relation.

Understandings of the Equals Sign: the Relational View

"Operational" is the word in the literature used to characterize these weak understandings. Roughly, an operational understanding involves viewing the equals sign as involving a performance of an operation. The authors contrast an operational understanding with a "relational" understanding, which is characterized in various ways. Although the authors vary in their meanings for "operational" and "relational," they are consistent in that an "operational" view always describes an incorrect or unproductive understanding, and a "relational" view describes a correct or productive understanding. I will adopt this terminology throughout the remainder of this paper. Due to space constraints, we only discuss characterizations of the "relational" view.

In the literature, the operational view is contrasted with the relational view. Authors tend to treat a relational view as anything that is not an operational view, but they are not explicit about this dichotomy and are sometimes imprecise or narrow about what they mean by a "relational" view. In all cases, the relational/non-operational view is what the authors endorse as the desired view for students to hold. Despite the attention that philosophers and mathematicians have paid to the nuanced meanings of the equals sign, math educators focus only on the misunderstandings of the equals sign. That is, they focus on students' operational understandings and do not consider the varied understandings within the relational characterization. As discussed, even within the mathematical community, there is a lack of consistency and clarity about what the equals sign means. While the nuances of the relational view vary across authors, what remains consistent is that the relational view involves viewing the equals sign as expressing an equivalence relation. Exactly what this equivalence relation is and what it applies to is not apparent or consistent in the literate.

Several authors characterize a relational view of the equals sign in such a way that is tantamount to expressing a relation between names of numbers. Denmark et al. (1976) and Kieran (1981) describe a relational view in a precise but narrow way: a student has a relational meaning for the equal sign when she sees whatever is on each side of the equals sign as names for the same number. Sáenz-Ludlow and Walgamuth (1998) and Oksuz (2007) do not clearly state that a relational view means "names for the same number," but do seem to imply it; Seans-Ludlow and Walgamuth (1998) describe a relational understanding as "quantitative sameness of two numerical expressions" and Oksuz (2007) describes it as a view that the equals sign is "a relationship expressing the idea that two mathematical expressions hold the same value" (p.3). Unfortunately, the authors do not define what "quantitative sameness" or "same value" means, but they indicate that the equals sign expresses a relation between symbols or signs by their use of the term "expressions". Notice that characterizing the equals sign as expressing a relation between signs or linguistic objects is a view that Frege originally espoused and then later rejected in favor of the notion of *sense*.

Some authors suggest a relational meaning of the equals sign as involving sameness as an *attribute* of expressions. For example, Oksuz (2007) refers to the expressions having the same "value" but does not elaborate further on what he means by "value". McNeil and Alibali (2005a), Knuth, E., Alibali, M., Hattikudur, S, McNeil, N., & Stephens, A (2008), and Seans-Ludlow and Walgamuth (1998) refer to "quantity" but in slightly different ways. McNeil and Alibali (2005a) say that the expressions actually *are* the "same quantity," Seans-Ludlow and Walgamuth (1998) refer to the "quantitative sameness of expressions," and Knuth et al. (2008) refer to the equals sign as "representing a relationship between two quantities. " Notice that McNeil and Alibali refer to the *expressions* as being the same quantity, whereas Knuth et al. do not explicitly refer to expressions and instead refer to the equals sign as a relation between quantit*ies* (plural). McNeil and Alibali's characterization is a bit odd - if an expression is a quantity, then different expressions (which is usually what is on either side of the equals sign) should indicate different quantities, yet they refer to the "same" quantity. Knuth et al's characterization is also a bit odd - if the equals sign is referring to *two* quantities, then what is it that is the same? None of the authors define what they mean by "quantity," nor do they explain what the equals relation *says* about quantities. Behr et al. (1980) are more explicit than the other authors about the relational meaning of the equals sign. They say that "the most basic meaning is an abstraction of the notion of *sameness.* This is an intuitive notion of equality which arises from experience with equivalent sets of objects. This is the notion of equality which we would hope children would exhibit" (p.13).

In other words, Behr et al. seem to be suggesting a meaning of the equals sign that alludes to the sameness of numerosity of equinumerous sets of objects, that perhaps, "3+4" refers to the cardinality of the set resulting from forming the union of a set of 3 elements with a set of 4 elements, and a sentence such as “3+4=5+2” has the meaning that such a set is equinumerous to the set resulting from the union of a set of 5 elements with a set of 2 elements.

Several authors (McNeil and Alibali, 2005 a and b; Byrd et al., 2015; Behr et al., 1980, McNeil et al, 2006; Kieran, 1981) ambiguously refer to the relational view of the equals sign as expressing “equivalence”. Unfortunately, the authors are not always clear about what particular equivalence the relation is on -- perhaps expressions or numbers -- nor are they clear about what the equivalence relation is. In fact, the authors do not explicate the relevance of the properties that make an equivalence relation an equivalence relation -symmetry, reflexivity, transitivity.

Kieran describes the relational view as an “equivalence view” of the equals sign and synonymous with “another name for.” She opens her paper with a quote from Gattegno (1974) that *“equivalence* is concerned with a wider relationship [than identity or equality] where one agrees that *for certain purposes* it is possible to replace one item by another. Equivalence being the most comprehensive relationship it will also be the most flexible, and therefore the most useful” (p.83, emphasis added). Gattegno is not contrasting equivalence with computational or operational understandings - instead, he is contrasting equivalence with identity. He is making the point that identity is a special *kind* of equivalence. In other words, he is emphasizing the difference between identity and equivalence, not treating them as one-and-the-same. Gattegno elaborates on what he means by “equivalence” by discussing an analogy between mathematical statements and natural language; he describes equivalence as a sort of linguistic replaceability that is a consequence of identity. In his view, “2x16” and “32” name the same thing and hence, are equivalent in the sense that in mathematical computations it is permitted to replace one with the other. He compares this sort of permitted replacement with replacing “he is on my right” with “I am on his left”. In other words, for Gattegno, equivalence is a consequence of identity that allows for linguistic replacement in certain contexts. This sort of permitted replacement -- the “equivalence” -- is what Gattegno suggests gives identity its power. Gattegno is not explicit about what this equivalence relation is, but he seems to hint that it is an equivalence relation between signs - e.g., “a” and “b” are equivalent if and only if “a” is replaceable by “b”. Gattegno hints at a relationship between viewing terms as “equivalent” and viewing them as “names for the same thing” - one is a consequence of the other. He explains that “4+1” and “5” are names for the same thing, and that therefore they are “equivalent” in the sense that they are interchangeable linguistically. Kieran, in citing Gattegno, does not seem to notice the relationship between “name for same thing” and replaceability. She instead muses that *another name for* is an equivalence relation on ordered *pairs* of numbers, which she calls “R”: “(a,b)R (c,d) iff a+b=c+d”. This relation R applies to statements like “4+5=3+6,” but not “4+5=9” or “9=9”. Moreover, it is not an equivalence relation on names - it is an equivalence relation on actual ordered pairs of numbers and only applies in narrow contexts. Moreover, it is odd that she seems to define this equivalence relation in such a way that it allows only for equations with two summands - why not instead say “aRb if and only if ‘a’ and ‘b’ name the same thing”?

Kieran is not the only author ambiguously using the word “equivalence”. Other authors also refer to the equals sign as expressing an “equivalence relation” but are unclear about what this relation is or what the relation is on (Knuth et al. 2008; McNeil & Alibali 2005; Byrd et al. 2015).

Despite the ambiguous characterizations of the relational understanding, we can tease out some significance of viewing the equals sign as an equivalence relation. It is important that students conceive of the equals sign in such a way such that it expresses an equivalence relation.

Further, having the properties of an equivalence relation is what allows for substitutability. Let us return to the “rule violations” that characterize an operational (non-relational) understanding:

	Rule Violations	**Rule Violated**
(i)	$5 = 2 + 3$	The “answer” comes to the right of the “problem”. Here, the “answer” is on the left.
(ii)	$2 + 3 = 4 + 1$	The “answer” should follow the “problem”. Here, the answer is “5,” but no answer is written.
(iii)	$5 = 5$	There needs to be a “problem”. Here, there is no problem.

Figure 1: three equations that students with operational views of the equals sign frequently reject

Viewing the equals sign as an equivalence relation accounts for a way of thinking in which (i), (ii), and (iii) are not rule violations, and, relatedly, the power of replaceability. Suppose = is an equivalence relation. So long as 2+3=5, it follows from symmetry that 5=2+3, in which case (i) is no longer a rule violation. Similarly, it follows from reflexivity that 5=5, in which case (iii) is no longer a rule violation. Concluding (ii) is a bit more involved but can be easily obtained through symmetry and transitivity: we have that 2+3=5, and by symmetry, 5=4+1. Hence, it follows from transitivity that 2+3=4+1. In other words, it is permissible in a mathematical context to use “5”, “2+3”, and “4+1” interchangeably - i.e. we can replace one term with another and have used the properties of equivalence to do so.

Discussion

The equals sign literature discussed above indicates that many students are *not* conceptualizing the equals sign as an equivalence relation. If someone were to view the equals sign as representing an equivalence relation, then the rule violations commonly discussed in the literature would not be rule violations; students would accept “5=2+3”, “2+3=4+1,” and “5=5” as true assertions. The most common misconception manifested itself in a rejection of statements like “2+3=4+1” with an acceptance of equations like “2+3=5+1=6” (Rule Violation (iii)). Renwick (1932) found this misconception amongst 8-14 year old girls of varying abilities. Oksuz (2007) found this misconception amongst middle school students. Fifty 5th graders and sixty 6th graders were asked what goes in the blank in “6+7=___+4”, and 38% of 5th graders and 24% of 6th graders answered with “13”. Behr et al. (1980), through performing individual interviews with 6-12 year old students, found that students viewed (i), (ii), and (iii) all as rule violations. When asked to give a definition of the equals sign, most children’s responses could be summarized as “when two numbers are added, that’s what it turns out to be”. In other words, students have a conception of the equals sign in such a way that it does not express an equivalence relation, and therefore do not have a mathematically normative or productive understanding.

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