How Peer Mentors Support Students in Learning to Write Mathematical Proofs

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We study how the mathematical beliefs and knowledge of peer mentors in a summer mathematics program influenced their efforts to help high school students learn to write proofs in number theory. Using Schoenfeld's framework for understanding decision making, we analyze interviews of three undergraduate student mentors for evidence of how their views of the role of proof, norms for proof writing, and mathematical knowledge for teaching informed their pedagogical decisions. We find that each mentor developed a distinctive approach to providing feedback on student work consistent with their own values, and present evidence that the success of each approach depended on the mentor's resources for interpreting student work.

Key words: Mathematical Proof, Mathematical Knowledge for Teaching, Number Theory

Research in undergraduate mathematics education has documented some common challenges associated with teaching students to identify and produce correct mathematical proofs. These challenges are often rooted in issues of what students consider to be a valid proof: students are often found to possess ritualistic or empirical proof schemes and derive conviction from the form of an argument rather than its analytic content (Harel & Sowder, 1998). When reading an argument and deciding whether it is valid, students frequently focus on surface features such as whether the presentation of the argument engenders a feeling of "making sense," or the extent to which the proof represents reasoning symbolically or verbally (Bleiler, Thompson, & Krajčevski, 2014; Selden & Selden, 2003). The process of constructing proofs also presents challenges: for example, in addition to knowing relevant mathematical facts, students must also develop strategic knowledge of when proof techniques or theorems are likely to be useful (Weber, 2001).

Recent research also reveals complications inherent in the notion of "correct" mathematical proof. A study of mathematicians' validations of an elementary analysis proof has suggested that there is no single standard for validity shared by all members of this community (Inglis, Mejia-Ramos, Weber, & Alcock, 2013). Moreover, mathematicians disagree on whether nonstandard uses of language in mathematical proofs constitute breaches of convention, and their judgments of potential breaches may be influenced by the context in which proofs appear (Lew & Mejia-Ramos, 2017). Those tasked with teaching students how to write proofs thus face a doubly difficult task: they must help students develop the interpretive frameworks and strategic knowledge necessary to read and produce proofs, while also facilitating their enculturation into a community whose norms are not well-defined and may vary depending on context.

In this study, we investigate the beliefs and knowledge that guide undergraduate student mentors' efforts to teach high school students in a summer number theory course to write proofs that conform to these mentors' perceptions of standard conventions for mathematical writing. We aim to contribute to the growing body of research on the mathematical and pedagogical resources entailed in teaching students to read and write proofs.

Background and Theoretical Framework

We assume the theoretical stance, suggested by Schoenfeld (2010), that people's teaching decisions can be understood on the basis of their goals, their beliefs and orientations, and their knowledge and resources. In the context of our study of peer mentors' decision making about how to help students learn to write proofs, we construe "goals" to include not only the broadly shared goal of helping students become effective mathematical writers, but also subgoals associated with teaching students to write proofs that can achieve specific purposes envisioned by the mentors, such as convincing other students of the validity of a claim or helping others understand why the claim is true. Prior research on proof has enumerated different roles that proof can play in mathematical learning and practice; these include strengthening the audience's certainty of the truth of a mathematical statement, explaining why a statement is true in terms of its connections to other known facts, developing a formal axiomatic system of concepts and theorems, and transmitting mathematical knowledge to other practitioners (De Villiers, 1990; Hanna, 2000). We hypothesize that mentors' goals in helping students learn to write proofs may align with some of the purposes suggested by this framework.

Within "orientations" we include mentors' beliefs about attributes that "good" or "correct" proofs should have, along with beliefs about how they can most effectively develop in students an appreciation for these attributes and habits that will help them write proofs that meet these standards consistently. Lai, Weber, and Mejia-Ramos (2012) found that mathematicians believe that pedagogical proofs, which serve primarily to explain and communicate mathematical results to an audience of students, should contain introductory and concluding sentences, should format major ideas so as to emphasize their importance, and remove redundant or extraneous information in order to minimize confusion. We hypothesize that since proofs that students produce in a number theory course often have the same purposes, mentors' beliefs about desirable attributes of proofs may align with these preferences. In addition, mentors may have linguistic norms for proof that align more or less with those of the mathematical discipline, such as those that forbid allowing a symbol to represent two different objects or that discourage stating entire definitions that are external to a proof (Selden & Selden, 2014).

We conceptualize "resources" as knowledge that mentors deploy in the work of teaching. Numerous studies have illustrated distinctions between the mathematical knowledge used in teaching and the mathematical knowledge that people use in everyday life and in non-teaching careers (e.g., Shulman, 1986; Ball, Thames, & Phelps, 2008). Mathematical knowledge for teaching (MKT) includes both subject matter knowledge, including specialized knowledge that helps teachers interpret mathematical thinking, vet problem-solving strategies and select representations of concepts, and pedagogical content knowledge, which entails understanding students' ways of thinking about mathematics and ways in which concepts can be presented in classroom settings (Hill, Ball, & Schilling, 2008). In the context of a proof-based number theory course, mentors' MKT might include knowledge that helps them interpret student proofs with unexpected features and approaches, and strategies for helping students identify a productive problem-solving approach without directly advising them on how to approach a problem.

Guided by this framework, we address the following research questions:

1. How do peer mentors' goals, orientations, and resources influence their pedagogical approaches in helping students learn to write number theory proofs?
2. How are these pedagogical approaches reflected in their evaluation and marking of hypothetical and actual student proofs?

Method of Study

Our study took place at a six-week summer mathematics program for high school students in the United States in 2018. The program included 63 students, of whom 35 were first-time participants taking a course in number theory. While the program is highly selective, admitting less than 20% of applicants, our initial interviews of study participants suggested that most students had not had extensive prior experience with mathematical proof beyond what some had encountered in U.S. regional and national mathematics contests.

On a typical day of the program, first-year students attended a lecture in the morning, took other classes in the afternoon, and participated in a four-hour homework session during the evening. During these homework sessions, students worked in “study groups” of three or four to prove theorems that would be covered in subsequent lectures. At the start of the program, each study group was assigned a peer mentor who supervised the evening sessions; the small groups and their peer mentor assignments remained stable throughout the program. In addition to attending lecture in the mornings and working on developing proofs of theorems in the evenings, students attended afternoon problem sessions, led by peer mentors, in which they discussed solutions to homework problems they had completed and foreshadowed upcoming content.

We chose to conduct a case study (Yin, 2013) of peer mentors’ approaches in helping students learn to write proofs because unlike the program faculty, who interacted with students primarily through lecture-based classes, the mentors had considerable opportunity to influence students’ views on proof and proof-writing through daily problem sessions, extended interactions during homework sessions, and their marking of each day’s completed homework. In addition, because peer mentors were typically undergraduate students in STEM disciplines, they were themselves in the process of learning to write and critique technical texts such as mathematical proofs; thus studying the work of mentors provided a unique opportunity to investigate the role of mathematical knowledge in teaching higher mathematics in a setting in which this knowledge was under active development.

To investigate how peer mentors supported first-year students in learning to write number theory proofs, we conducted interviews of four mentors (Table 1) during the second week of camp and during the final week of camp. During the initial interview, we asked questions about mentors’ beliefs about the purposes of proof in the context of the number theory course, what it means for a proof to be correct or incorrect, and how participants supported their students in learning to identify correct proofs.

Table 1: *Participating Mentors and Demographic Information*

Mentor	**Age and Ethnicity**
Linda	17, Asian/Pacific Islander
David	19, White
Nina	18, Asian/Pacific Islander
Nathan	18, African American

Following are a few of the questions we asked in the initial interviews:

1. In the number theory course, proofs are given for most of the facts discussed in class. Why do you think the class does this?
2. What are some characteristics of good mathematical proofs?

3. When a student submits a proof, how do you decide whether the proof is correct?
4. How do you support your students in learning to construct correct proofs, and distinguish correct proofs from incorrect ones?

In the final interviews, we repeated some questions from the initial interview to track possible shifts in mentors' beliefs about proof and about teaching students to write proofs. We also asked each participant to read and mark a hypothetical student's proof of Euclid's lemma (that if a, b, and c are integers such that a divides bc and a is relatively prime to b, then a divides c). We also asked participants to explain their actual markings of several of their own students' proofs; this allowed us to gain insight about how participants' beliefs and MKT informed their approaches to the everyday work of mentoring. In these final interviews, we used a tablet to display scans of the hypothetical student proof and actual student proofs so that we could record mentors' markings on proofs as well as their spoken comments.

We transcribed audio from the initial and final interviews for each of the four mentors in our study. We analyzed transcripts using thematic analysis (Braun & Clarke, 2006) to identify themes in participants' beliefs about purposes of proof and reasons for learning to write proofs, beliefs about features that influence the quality or validity of a proof, and mathematical and pedagogical knowledge that was relevant to the work of helping students learn to write proofs.

Results and Analysis

In this section we discuss our analysis of our interviews of three of the four mentors; we selected these three cases because each revealed themes not readily visible in the other cases. For each case we include some extracts from our interviews with the peer mentor that shed light on their goals and orientations with respect to the teaching of mathematical proof; we also include some observations about their practices in marking proofs, as evidenced by their responses to the hypothetical student proof task and their discussion of their actual students' marked proofs.

Proof as Disciplinary Activity: The Case of Nina

When asked in her initial interview why the number theory course focuses on developing proofs of mathematical theorems rather than simply presenting facts and computational strategies, Nina discussed the role of proof in explaining how and why mathematical ideas work:

> I think it does this because the class is really focused on not so much accumulating facts and information, which you are doing as you go through the problem sets, but also understanding why each one of them works the way they do. We start with intuitive - sort of, quote, "simple" statements such as n times zero equals zero, and they're statements that we often take for granted. So when you dive into the axioms behind those and how they really work, you understand math from one different perspective, and then also a deeper perspective. You have a more solid understanding of it.

This and other responses from Nina suggested that one of her goals was to help students learn how to write proofs that would shed insight on conceptual underpinnings of and connections among mathematical ideas. The theme of proof as an avenue for deepening mathematical knowledge recurred in many of Nina's answers during the initial interview.

When asked about characteristics of "good" proofs, Nina highlighted the importance of developing an argument that is rigorous; when asked to clarify what "rigorous" meant, she described a rigorous proof as one that "explain[s] every step thoroughly and carefully," and that peers can understand without difficulty. She suggested that a proof should have "eloquence," observing that some of her students often used colloquial language or wrote in incomplete sentences. Finally, she noted the importance of validity, which she characterized as not assuming the conclusion, taking incorrect logical steps, or performing steps that could not be justified in terms of facts already proven. She also noted that a proof should minimize unnecessary steps. We view these norms for mathematical proofs as orientations that Nina might apply to the work of marking proofs; we note that some of these norms (such as omitting unneeded steps) are consistent with those described by Selden and Selden (2014) and Lew and Mejia-Ramos (2017).

Nina's initial interview also offered insight into her orientations regarding her role in helping students improve at proof-writing. She described a practice, shared by most mentors in the camp, of assigning "redos" and "rewrites" for proofs deemed to be inadequate. Nina characterized the distinction between a "redo" and a "rewrite" as based on the depth of errors in a proof; while a proof that contains a major error might warrant a "redo," a proof that contains a valid argument but has some writing errors (such as failing to introduce variables) might only receive a "rewrite." When asked about the pedagogical purpose of assigning redos and rewrites, Nina said:

> I think the point of a rewrite is to show them how to make their proof better, and show that you're missing a few steps here, and that *if you practice rewriting this, you'll write better proofs in the future*. I try to stress to my campers that it's not a bad thing to get a rewrite or a redo, it's not like you failed an assignment or you did poorly, got a bad grade. It's just that here's an opportunity for you to fix this proof, *and then next time you'll write an even better proof from there on*. [emphasis ours]

Thus for Nina, assigning redos and rewrites served as an opportunity to reinforce normative proof-writing practices for students.

During the final interview, we asked Nina to review two of her students' proofs that multiplying each element of a complete residue system modulo *n* by a unit produces another complete residue system. Both students had written proofs using similar approaches, but Nina had marked one proof correct while assigning the other a redo. When asked about this discrepancy, Nina explained that while the first student's writing suggested a sound understanding of the approach, the second student's writing did not:

> I think again, it's little things - you missed a word here, that shows that perhaps you're kind of writing things based off what you remember from presentation, but you're not fully there. It's really hard to define. It's subtle distinctions here. Two students can write almost the same amount of text, and one can just show that they understand better than the other did, just by the words they've selected and the way they have presented their proof.

This excerpt suggests one type of knowledge that Nina used in marking proofs; while a reader not concerned with individual students' understanding might have marked both proofs correct, Nina used her knowledge of content and students to discern the depth of a student's understanding of an argument. When a student's writing suggested a lack of understanding in tension with Nina's goals for students' proving activity, she asked the student to revisit it.

Proof as Persuasion: The Case of David

David's responses to questions about the purpose of proof in the number theory course focused on the roles of proof in verifying and communicating mathematical results. In both of his interviews, he showed commitment to the notion of a proof as a persuasive essay, and suggested that skills students developed in the number theory course could prove valuable elsewhere:

> I think by starting from the bare minimum, like the axioms, and building up on that, you're learning to justify everything you say, and that's a skill you need everywhere in life. If you're

> writing an essay, a persuasive essay, everything you say has to follow from some basic assumption, and you have to justify everything; otherwise a reader's not necessarily going to be persuaded. I think that's true in basically any field.

David's answer suggested that one of his goals was to help students learn to write proofs that could persuade peers of the truth of a mathematical claim.

In discussing norms for "good" mathematical proofs, David stated that a successful proof should be something that a layperson could understand, given a sufficient understanding of the problem under discussion. He noted the importance of using complete sentences and including explanation for each step of a proof. When asked how he approached the task of validating a proof, David suggested the notion of a skeptical reader who might identify holes in an argument:

> Everything could be technically correct, but if there isn't explanation behind each step, then it doesn't have value to somebody else. … If they aren't able to convince someone of something entirely, then I don't think it's correct. If I can, as a reader, think "oh, what about this case?" and they haven't addressed that, then I don't think it's a correct proof, because they have to be able to irrefutably convince you of something.

This suggests that some of David's beliefs about proof quality may have oriented him toward focusing on the flow of students' reasoning in proofs and their consistency in justifying steps and addressing all cases of a problem. However, we observed instances in which David's curricular knowledge of the number theory course may have imposed some limitations on this. In their proofs of the theorem that if b is nonzero and $a = bq + r$, then $\gcd(a, b) = \gcd(b, r)$, David's students used the fact that if $\gcd(a, b) = d$, then $\gcd(a/d, b/d) = 1$; they also used the fact that if $\gcd(a, b) = d$, then $\gcd(ak, bk) = dk$. The first of these facts can be proven using a relatively straightforward argument; the other is typically proven using Bézout's identity, for which this theorem is often a building block. Because David did not recognize that these facts would be systematized later in the curriculum (or was willing to accept these claims based on intuitive reasoning), he did not object to their use. When we asked David about these proofs, he initially judged the proofs to be correct, but revised this judgment when presented with the fact that the students' subsidiary claims about greatest common divisors had not yet been formalized.

Proof as Opportunity for Assessment: The Case of Nathan

Nathan framed many of his observations about the power and importance of mathematical proof in contrast with his experiences with traditional schooling. When asked why the summer program provides students with such extensive experience with mathematical proof, Nathan discussed the development of students' intellectual agency:

> I think having them write up their ideas and work on their own to prove theorems is very important, in the sense of not having a higher power or a teacher … do it for them. … If you're working with others who are trying to understand the same way you are and you guys are bouncing ideas off of each other - even if it's wrong at first, even if your proof is wrong, you work with each other to try to reach this conclusion of why this works, and how it's true. I think that's how people did it before there was anybody to really help them understand something - they'd bounce each other's ideas off of one another and come to a conclusion that was right and made sense to them.

In discussing his orientations regarding proof quality, Nathan consistently focused on two attributes of "good" proofs: rigor (which he characterized as attending to all of the details in a proof's reasoning) and clarity. His discussion of his teaching approach during both the initial and final interviews suggested an iterative approach to vetting students' proofs: when faced with a claim in a student proof whose justification was unclear or lacked rigor, he would engage in a one-on-one conversation with the author and ask questions to assess their understanding of the reasoning. If the student demonstrated sufficient understanding of the reasoning in the proof, Nathan would make a minor suggestion as to how the student might better convey this reasoning rather than assigning a more extensive rewrite. Thus Nathan used his knowledge of content and teaching to identify ways to honor students' agency in presenting and explaining their own reasoning – a goal he clearly valued – while maintaining standards for mathematical rigor.

Discussion

The cases of Nina, David, and Nathan illustrate ways in which peer mentors' goals and orientations might guide their norms for the proofs that students create as well as their approaches in helping students learn to create proofs consistent with these norms. They also suggest ways in which various facets of mentors' MKT might afford or constrain opportunities to make progress toward their self-defined teaching goals in ways consistent with their beliefs about mathematical proof and about teaching and learning.

The study took place in a setting in which students and mentors have co-constructed a distinctive set of norms for proof validity and quality that may not be consistent with those of the professional mathematicians who direct the program (Patterson & Cui, 2017). In particular, we hypothesize based on results from this and our previous study that peer mentors in this setting may have higher standards than most of the mathematical community for the amount of detail students must provide when justifying claims; for example, demands that students cite axioms for the integers, such as the commutative and distributive properties, fall off much later than they do in most number theory courses. In this study, however, we see that demands on students' justifications may originate from different pedagogical beliefs and intentions. While Nina's standards and practices seemed focused on maintaining the integrity of disciplinary norms for proof writing, David's emerged from a view of proof as argumentation, a practice that he viewed as transferable across disciplines. While Nathan had similarly stringent standards, he appeared to offer students greater flexibility in how they met these standards, and seemed interested in maintaining them in order to maximize students' opportunities to develop and demonstrate understanding. All three mentors responded in similar ways to an interview task that asked them to mark a hypothetical student proof, suggesting that they had similar norms for proof quality and comparable consistency in enforcing these norms; however, their motivations for enforcing these norms appear to be more diverse than we had originally hypothesized.

Limitations of Study and Next Steps

In this study we analyzed peer mentors' beliefs and practices for teaching students to write proofs in number theory. We do not yet understand how students interpret their mentors' feedback about the proofs they write, how these interpretations inform the iterative development of students' own beliefs about proof validity, or with how much fidelity students adopt the beliefs and norms of their mentors. We also hesitate to make broad inferences about mentors' MKT based on their responses to interview prompts, since some questions involved proofs that they had marked three to four weeks prior to the final interviews. Furthermore, the peer mentors work in an environment in which time for marking papers is scarce; failures to identify errors in proofs may be due to time constraints rather than gaps in mathematical knowledge.

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