A Comparison of Frameworks for Conceptualizing Graphs in the Cartesian Coordinate System

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The use of the Cartesian Coordinate system (CCS) pervades secondary and tertiary mathematics curriculum, as the dominant convention for displaying graphs of functions. The CCS in two dimensions may be framed as a conceptual blend of two number lines and a Euclidean plane (Lakoff & Núñez, 2000). Within the concept of a number line is a conceptual metaphor uniting numerical values with points on a line. While such a description of the CCS may describe a shared understanding of the convention among the mathematics community, it may not account for the ways in which individual students interpret graphs presented in the CCS. Other theories, such as David et al.'s (2017) constructs of value-thinking and location-thinking, have been proposed to account for students' graphical interpretations. In this paper, I outline these two ways of framing conceptions of graphs, the uses of each framework, and their relation to each other.

Keywords: Cartesian Coordinate System, Graphing, Conceptual Blend, Conceptual Metaphor, Value-Thinking and Location-Thinking

Across numerous mathematics courses at the secondary and undergraduate level, students are asked to interpret and reason with graphs that are represented in the two-dimensional Cartesian coordinate system (CCS). In the U.S., the CCS is typically the first coordinate system in which students are expected to graph points (5.G.A.1-A.2) and the standard coordinate system used in curriculum from Algebra through Calculus (e.g., Stewart, 2012). The CCS, like other coordinate systems, follows certain conventions. In two dimensions, the Cartesian plane consists of two axes with specified units that meet at a right angle. Pairs of values are represented given distances from the intersection of these axes, referred to as the origin. Due to its fundamental role in the teaching and learning of mathematics at the secondary and undergraduate level, researchers have examined ways in which both individuals and the mathematics community use, reason with and interpret graphs in this coordinate system. Through various modes of research from different perspectives, several theoretical frameworks have been proposed to explain some ways in which graphs are understood in the CCS.

In this paper, I will describe two theoretical frameworks from different theoretical traditions, explain how they may be used, and offer some examples of how researchers have used these to frame their data analysis. I will also compare the purposes and benefits of adopting and utilizing each of these frameworks. One framework comes from the work of Lakoff and Núñez (2000), whose perspective offers insight into the underlying cognitive structure of the CCS as developed and used by the mathematics community. In their description, the CCS relies on a conceptual metaphor of numbers as points to understand. While Lakoff and Núñez's (2000) framework offers one view of the CCS as a conventional system, rooted in an embodied cognition perspective, their theory may not readily apply when describing the ways in which individuals may interpret graphs represented by such a system. For instance, instructors teaching students content that includes graphs in the CCS may use different understandings than those proposed by Lakoff and Núñez (2000). Furthermore, the way in which students interpret graphs presented to them in their courses may differ from the mathematics community as well as their instructors. Thus, I will also describe David, Roh, and Sellers (2017) framework to characterize students' graphical interpretations. Their framework also recognizes the role of both values and locations of points in interpreting graphs, namely that students may attend to one aspect of points rather than the other in their interpretations. Both David et al.'s (2017) framework, as well as Lakoff and Núñez's (2000) theory offer valuable insight into cognition related to graphs. The adoption of one theoretical frame for conceptualizing graphs rather than another ought to be guided by a researcher's purposes.

Conceptual and Ideational Mathematics

To frame my discussion of the content and purpose of these frameworks for studying conceptions of graphs, I adopt two considerations offered by Schiralli and Sinclair (2003) in their commentary on the work of Lakoff and Núñez (2000). The first is the distinction they make between conceptual mathematics and ideational mathematics. In their explanation, Lakoff and Núñez (2000) offer a description of *conceptual mathematics*, which refers to the discipline of mathematics as a collective subject matter, negotiated by participants in the mathematics community who hold a shared meaning. In contrast, they use the term *ideational mathematics* to refer to the ways in which individuals interpret or reason about conceptual mathematics. Ideational mathematics includes the ways that mathematicians may use and conceptualize mathematical ideas and the ways students may interpret and understand ideas. In defining these terms, Schiralli and Sinclair seek to clarify whether the mathematical concept to be studied is shared knowledge in the field of mathematics or lies in the mind of an individual engaged in mathematical thinking. Schiralli and Sinclair (ibid) also emphasize that the way in which a particular group or individual is engaged with the mathematics, "whether one is *learning, doing,* or *using* mathematics" may influence their cognitive processes and should be considered (p. 81). For the purposes of discussing the theoretical frameworks in this paper, I follow these two considerations posited by Schiralli and Sinclair (ibid): I situate each framework and its use based on (1) "which mathematics" aims to be studied and (2) the nature of the goals of the individual or group conceptualizing the mathematics. These considerations of Schiralli and Sinclair (ibid) help to make explicit certain underlying assumptions within each theoretical framework as well as offer insight into how these frameworks may serve researchers in their purposes of investigating various conceptions of mathematical ideas.

Cartesian Plane as a Conceptual Blend

Lakoff and Núñez (2000), who operate from a perspective of embodied cognition, view mathematical thinking as fundamentally rooted in humans' sensorimotor experiences, influenced by their neural biology. In their work to describe "where mathematics comes from," Lakoff and Núñez (ibid) seek to reveal and untangle the underlying cognitive structures that serve as the foundation of mathematics as a discipline. Their framework derives from a method of "mathematical idea analysis," a linguistic approach in which they uncovered underlying metaphors from the language used in central concepts in mathematics (Schiralli & Sinclair, 2003). Thus, the frameworks they propose for making sense of the cognitive structure of mathematical ideas describe *conceptual mathematics* used by mathematicians in doing mathematics.

Relative to graphs in the Cartesian Coordinate System, Lakoff and Núñez (2000) describe the Cartesian Plane as a conceptual blend, a combination of conceptual domains. In their description, number-lines make up the axes of the Cartesian Plane, which rely on the conceptual metaphor "Numbers are Points on a Line." Lakoff and Núñez (ibid) describe conceptual metaphors as a cognitive tool to make concrete concepts which are inherently abstract, such as those in mathematics. In a conceptual metaphor, an object is mapped from a source domain to another object in a target domain in such a way that preserves inferences. In the Numbers are Points on a Line" metaphor, numbers are the target, abstract domain described by points on a line, a more concrete concept. Table 1 contains Lakoff and Núñez's (ibid) description of the key correspondences in the "Numbers are Points on a Line" metaphor.

Table 1. Numbers are Points on a Line (for Naturally Continuous Space) (Lakoff & Núñez, 2000, p. 279)

Source Domain Points on a Line		*Target Domain* A Collection of Numbers
A Point *P* on a line	→	A Number *P'*
A Point *O*	→	Zero
A point *I* to the right of O	→	One
Point *P* is to the right of point *Q*	→	Number *P'* is greater than Number *Q'*
Point *Q* is to the left of point *P*	→	Number *Q'* is less than Number *P'*
Point *P* is in the same location as point *Q*	→	Number *P'* equals number *Q'*
Points to the left of *O*	→	Negative numbers
The distance between *O* and *P*	→	The absolute value of number *P'*

In this metaphor, points on a line are the source domain, the concrete object, to which a collection of numbers is mapped. Ordering is one of the inferences preserved in this mapping, with points to the left defined as numbers with smaller values. Through what they refer to as the "Number-Line Blend," new objects are created which they refer to as number-points, at once numbers and points on a line.

Moving from one to two dimensions, the Cartesian Plane is described by Lakoff and Núñez (2000) as comprised of a conceptual blend. A conceptual blend, distinct from a conceptual metaphor, refers to a blending of "two distinct cognitive structures with fixed correspondences between them" (Lakoff & Núñez, ibid, p. 48). Table 2 shows the correspondences that comprise the Cartesian Plane Blend.

Table 2. The Cartesian Plane Blend (Lakoff & Núñez, 2000, p. 385)

Conceptual Domain 1 Number Lines		*Conceptual Domain 2* The Euclidean Plane with Line X Perpendicular to Line Y
Number line *x*	↔	Line X
Number line *y*	↔	Line Y
Number *m* on number line *x*	↔	Line *M* parallel to line *Y*
Number *n* on number line *y*	↔	Line *N* parallel to line *X*
The ordered pair of numbers (*m*, *n*)	↔	The point where *M* intersects *N*
The ordered pair of numbers (0, 0)	↔	The point where *X* intersects *Y*
A function *y*=f(*x*); that is, a set of ordered pairs (*x*, *y*)	↔	A curve with each point being the intersection of two lines, one parallel to *X* and one parallel to *Y*
An equation linking *x* and *y*; that is, a set of ordered pairs (*x*, *y*)	↔	A figure with each point being the intersection of two lines, one parallel to *X* and one parallel to *Y*
	↔	

The solutions to two simultaneous equations in variables x and y	The intersection point of two figures in the plane

In this conceptual blend, the cognitive structures of number lines and the Euclidean plane are combined. Each element of one domain combines with an element from the other domain. For instance, the x-axis in the CCS is a blend of both a number line '*x*' as well as a Line 'X' in the Euclidean plane. Similarly, points in the CCS are at once ordered pairs (m, n) and locations of intersections of two lines related to m and n, parallel to the y-axis and x-axis respectively.

Use of Cartesian Plane as Conceptual Blend Framework

Describing the mathematical use of number lines as relying on a conceptual metaphor and the Cartesian Plane as a conceptual blend may offer insight into the emergence of these conventions in the development of the field of mathematics. Namely, the mental act of ascribing geometric notions of locations and distances offers a powerful conceptual tool to conceptualize abstract ideas of number, ordered pairs, and functions. However, this framework characterizes ideas that have developed into shared meanings within the mathematical community, rather than individual differences in working with such ideas.

Although the nature of Lakoff and Núñez's (2000) framework is designed to characterize conceptual mathematics, the construct of a conceptual metaphor offers a lens to consider the ideational mathematics of individuals. For instance, Font, Bolite, and Acevedo (2010) investigated the metaphors that Spanish high school instructors used in their classrooms while teaching graphs of function. In their study, Font et al. (2010) were interested in investigating instructors' *ideational mathematics* while engaged in the act of teaching. They found that instructors used various metaphors to communicate properties of graphs. These metaphors included the graph as a path, orientational metaphors, and object metaphors, in addition to the ones identified by Lakoff and Núñez's (2000) description of *conceptual* mathematics related to graphs. Furthermore, instructors were found to be unaware of their use of language related to these metaphors in their instruction. When asked to consider their own metaphorical language, instructors commented that their purpose in using it was to support their students in understanding a certain principle. While this study examined how instructors interpret and use ideas related to graphing while teaching, other studies have focused on characterizing *students'* ideational mathematics.

Interpreting Graphs via Value-Thinking or Location-Thinking

In contrast with a perspective of embodied cognition, David et al.'s (2017) framework to characterize conceptions of graphs is situated in a constructivist perspective. This framework, shown in Table 1, details two ways *students* may interpret aspects of graphs, referred to as *value-thinking* and *location-thinking*. This framework was developed to describe phenomena that were observed in undergraduate students' interpretations of graphs in the CCS in the context of the Intermediate Value Theorem (IVT) (David et al., ibid). To be clear, their framework was not a priori theory; rather, this framework emerged from their data analysis (David et al., ibid). In this framework, if a student attends to the pairs of *values* that points represent, this way of thinking is referred to as *value-thinking*. On the other hand, if a student focuses on the *location* of points in the Cartesian plane, this way of thinking is referred to as *location-thinking*.

Table 3. Comparison of Characteristics of Value-Thinking and Location-Thinking (David, Roh, & Sellers, 2017, p. 96)

<table>
<tr><th colspan="2"></th><th colspan="2">Value-Thinking</th><th colspan="2">Location-Thinking</th></tr>
<tr><th colspan="2"></th><th>Interpretations</th><th>Evidence</th><th>Interpretations</th><th>Evidence</th></tr>
<tr><td rowspan="3">Aspects of a Graph</td><td>Output of Function</td><td>The resulting value from inputting a value in the function</td><td>▪ Labels output values on output axis
▪ Speaks about output values</td><td>The resulting location in the Cartesian plane from inputting a value in the function</td><td rowspan="3">▪ Labels outputs on the graph
▪ Labels points as outputs
▪ Speaks about points as a result of an input into the function (e.g., “an input maps to a point on the graph”)</td></tr>
<tr><td>Point on Graph</td><td>The coordinated values of the input and output represented together</td><td rowspan="2">▪ Labels points as ordered pairs
▪ Speaks about points as the result of coordinating an input and output value</td><td>A specified spatial location in the Cartesian plane</td></tr>
<tr><td>Graph as a Whole</td><td>A collection of coordinated values of the input and output</td><td>A collection of spatial locations in the Cartesian plane associated with input values</td></tr>
</table>

These two ways of thinking characterize students’ interpretation of graphs. The framework explains each way of thinking by detailing how a student engaged in that way of thinking thinks about three aspects of a given graph: outputs of the function, points on the graph, and the graph as a whole. Each of these aspects of graphs is described from the perspective of a researcher using conventional interpretations of the Cartesian coordinate system. The output of a function is conventionally represented as a magnitude of length in the direction of the y-axis. A point conventionally represents a pair of both input and output values, located a distance of the input value to the right of the origin, and a distance of the output value above the origin. Conventionally, a graph as a whole represents the set of all ordered pairs that satisfy the equation of the function. The framework also describes observable evidence indicative of thinking about aspects of the graph in a particular way. Using these descriptions of observable evidence in the framework, students’ words, gestures, and markings on the graph can be used to characterize their way of thinking about graphs as either value-thinking or location-thinking.

Value-Thinking

In this framework, value-thinking refers to an attention to the values represented by a point in Cartesian space. Students whom David et al. (2017) classified as engaged in value-thinking treated outputs as values associated with corresponding input values. These students may have indicated their thinking by labeling output values on the output axis, or speaking about output *values*. In their description, students engaged in value-thinking think of points as coordinated pairs of input and output values. These students may indicate this way of thinking by labeling points as ordered pairs, and speaking of simultaneous pairs of values when referring to points on a graph. Thus, students engaged in value-thinking treat graphs as a collection of points, each of which represents a pair of input and output values.

Location-Thinking

In contrast, location-thinking refers to an attention to the locations of the points in space. Students whom David et al. (2017) classified as engaged in location-thinking treated points on the graph as outputs, confounding outputs of the function with points on the graph. These students may have indicated that they were thinking in this way by referring to points solely as outputs or describing the output of a function as the location of the graph itself (e.g., “each input is mapped to a point on the graph”). Additionally, students engaged in location-thinking may label a point with an output value only, thus placing the output label at a point, rather than on the output axis. Thus, students engaged in location-thinking treat graphs as a collection of points that represent locations in the plane that correspond with input values.

Use of Value-Thinking and Location-Thinking Framework

To highlight the distinction between value-thinking and location-thinking, consider the two examples of sample student labeling on the same graph indicative of each of these ways of thinking, shown in Figure 1.

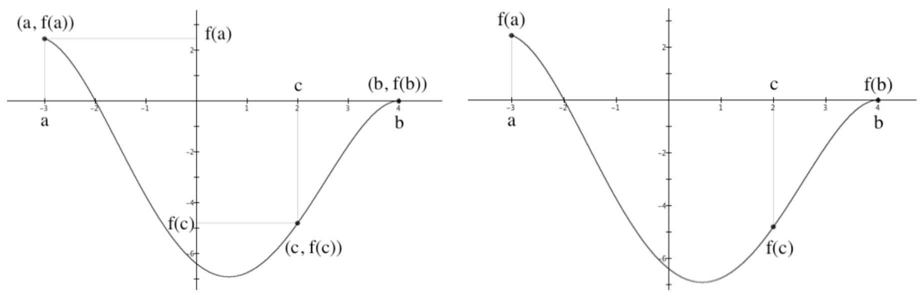

Figure 1: Example labels indicative of value-thinking, left, or location-thinking, right. (David et al., 2017 p. 97)

The labels on the graph in Figure 2, left, may indicate value-thinking. In this graph, output values are labeled on the output axis, and points are labeled as ordered pairs. In contrast, the labels on the graph in Figure 2, right, may indicate location-thinking. Output labels are not placed on the output axis but rather at the locations of points. Consequently, points are not labeled as ordered pairs but solely as outputs. While a student’s gestures and words should be examined in addition to the labels on a graph, these examples highlight distinctive characteristics of value-thinking and location-thinking.

David et al.’s (2017) framework emerged from analysis of a data set from interviews of nine undergraduate math students who were asked to evaluate and interpret statements related to the Intermediate Value Theorem using graphs. Their final coding scheme involved classifying students as engaged in value-thinking or location-thinking throughout episodes of their interviews. In their later work, David et al. (2018) report details of a student, Zack, whose thinking was characterized as location-thinking. See his graph labels in Figure 2.

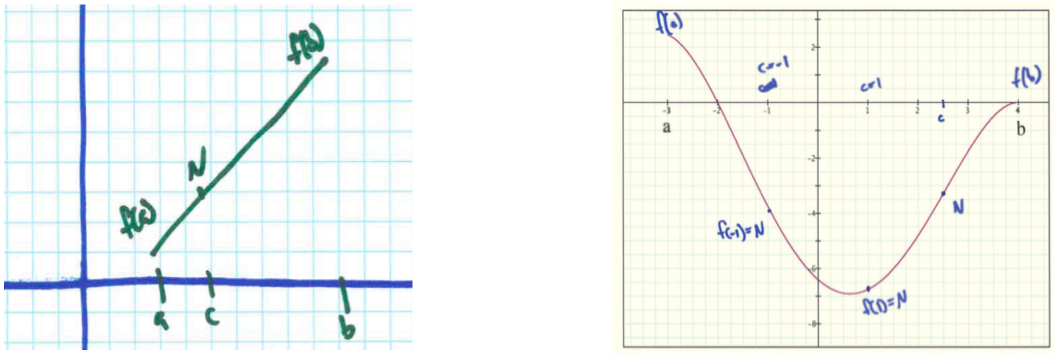

Figure 2. Zack's labels of points as outputs, a common characteristic of location-thinking (David et al., 2018)

David et al. (2018) point to several pieces of evidence support the claim that Zack was engaged in location-thinking when reasoning with these graphs and statements related to the IVT. First, Zack placed output labels at locations on the graph, rather than on the *y*-axis and referred to the endpoints of the graph as "*f*(*a*)" and "*f*(*b*)." Additionally, Zack labeled *N*'s between *f*(*a*) and *f*(*b*) along the graph of the function, rather than along the *y*-axis. In addition to the context used in David et al.'s (2017) study, this framework may also be applied by researchers to characterize student thinking in other contexts. Instructors may even find such a framework useful in attending to their students' reasoning when teaching graphs of functions. In my view, this framework best supports the goal of characterizing the thinking of students engaged in the learning of mathematics.

Conclusion

The study of the conceptions and uses of graphs in the Cartesian Coordinate System is a valuable line of research in mathematics education. In this paper, I have compared two such theoretical frameworks related to the study of graphing in mathematics in terms of its relation to conceptual or ideational mathematics and the activity of those engaged with the mathematics at hand. By framing the Cartesian plane as a conceptual blend built on metaphors, Lakoff and Núñez's (2000) framework supports researchers in uncovering the cognitive processes involved in considering graphs in the CCS within the domain of conceptual mathematics. By extension, researchers have begun to use their metaphorical framing to capture ways in which instructors conceptualize graphs while teaching (Font et al., 2003). Characterizing students' ways of interpreting graphs (their ideational mathematics), David et al.'s (2017) framework of value-thinking and location-thinking highlights previously undocumented phenomena in students' graphical interpretations. Both of the theoretical frameworks illustrated in this paper acknowledge two aspects of points on graphs in Cartesian coordinates: the values represented by points and the locations of these points spatially. The way in which these frameworks view this duality differs due to the perspective adopted. For Lakoff and Núñez (2000), a practitioner of mathematics uses this duality, even if subconsciously. From the perspective of David et al. (2017), the extent to which students conceptualize this duality varies; in fact, students may be more likely to focus on one aspect of a point rather than both simultaneously. Going forward, researchers should take careful theoretical consideration in deciding how to frame investigations of graphing. Such attention may yield extensions of the current frameworks, further delineations of these ways of thinking, or other characteristics that have yet to be identified. In this way, theory on graphing as part of mathematical activity will continue to be built and refined.

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