Teachers' Reasoning with Frames of Reference in the US and Korea

Surani Joshua, Arizona State University
Hyunkyoung Yoon, Arizona State University
Patrick W. Thompson, Arizona State University

Our theory of what entails a conceptualized frame of reference is explained, along with items and rubrics designed to illuminate how teachers do or do not reason with frames of reference. We gave 551 teachers in the US and Korea frame of reference tasks, and coded the open responses with rubrics intended to rank responses by the extent to which they demonstrated conceptualized and coordinated frames of reference. Our results show that our theoretical framework is useful in analyzing teachers' reasoning with frames of reference, and that our items and rubrics function as useful tools in assessing teachers' meanings for quantities within a frame of reference.

Keywords: Frames of Reference, Mathematical Meanings, Secondary Teachers, Quantitative Reasoning

A frame of reference is an organizing tool most familiar in physics, yet it is also applicable to any mathematics task that involves quantities, or measurable attributes of objects (Thompson, 1993). Every time a person thinks about a quantity, its meaning is only fully understood within the frame of reference within which it was measured. To say a plane is flying at 35,000 feet only has meaning when we know height was measured in a frame where the reference point is sea level; to say a ball's free fall velocity changes by -9.8m/s/s only has meaning when we know that acceleration was measured within a directionality where the measurements are always away from the center of the Earth.

If professional development programs and education researchers wish to address issues with how teachers help their students with the mathematics they teach, we first need more nuanced information about the teachers' own understandings of the mathematics. Many current assessments that focus on mathematical knowledge for teaching (Hill, 2005) categorize teachers' MKT by whether or not they can give normatively correct answers to tasks. Project Aspire wished to take an alternate approach by analyzing teachers' responses by what those responses told us about the teacher's current meanings (Thompson, 2016), and to compare different meanings by how productive they might be for helping students to develop coherent meanings. We did so by writing items and rubrics and analyzing responses from over 500 teachers in the US and Korea. Our work can be connected to critiques of the deficit model (Bak, 2001), in that we are interested in identifying what teachers *do* understand, in whatever ways they do.

In this work, we draw on data from part of an assessment that was developed to analyze teachers' mathematical meanings for frame of reference. The research question for this analysis is: *In what ways do teachers reason about quantities within frames of reference on our tasks?*

Past Literature and Theoretical Perspective

When we first began to write about teacher responses to frame of reference items (Joshua, Musgrave, Hatfield, & Thompson, 2015), our search of math education and physics education literature revealed no cognitive definitions of frame of reference. By 'cognitive definition' we mean a definition of what mental actions a student must engage in in order to use a frame of reference productively to solve tasks. Instead, the definitions we found in both textbooks and academic articles referred to physical objects, such as "a set of rigidly welded rods" (Carroll &

Traschen, 2005), “a set of observers” (de Hosson, Kermen, & Parizot, 2010), or “a coordinate system and a clock” (Young, Freedman, & Ford, 2011) among others. Several studies looked at ways in which students struggled with frame of reference tasks (Bowden et al., 1992; Trowbridge & McDermott, 1980) or reported results of interventions meant to improve performance on frame of reference tasks (Monaghan & Clement, 1999; Shen & Confrey, 2010), and one identified common student misconceptions about frames of reference (Panse, Ramadas, & Kumar, 1994). None gave a clear cognitive definition of frame of reference, which we concluded was needed to have a productive conversation about student or teacher reasoning on frame of reference tasks.

When we speak about a person who has fully conceptualized a frame of reference, the frame of reference itself is not the primary object of consideration. Rather, the person is using one or more frames of reference as a systematic way to think about and organize the measures of quantities and their meanings, as well as the quantitative relationships between those quantities. This places our constructs of conceptualizing and coordinating frames squarely within the domain of quantitative reasoning. This clarification guided our eventual definition:

> An individual can think of a measure as merely reflecting the size of an object relative to a unit or he can think of a measure within a system of potential measures and comparisons of measures. An individual conceives of measures as existing within a *frame of reference* if the act of measuring entails: 1) committing to a unit so that all measures are multiplicative comparisons to it, 2) committing to a reference point that gives meaning to a zero measure and all non-zero measures, and 3) committing to a directionality of measure comparison additively, multiplicatively, or both. […] An individual is coordinating two frames of reference if she conceives each frame as a valid frame, stays aware of the need to coordinate quantities’ measures within them, and carries out the mental process of finding a relation between the frames while keeping all relative quantities and information in mind. (Joshua et al., 2015)

We wish to emphasize that we are certainly not claiming that people explicitly say to themselves “I have decided to commit to a ____.” For most people these commitments are made implicitly, and are only observable indirectly by looking at how individuals reason through tasks and inferring the presence or lack of commitments that explain their responses. Our theory therefore functions as an explanatory framework for how people think about quantities.

Methodology

From 2012 to 2015, the Project Aspire team created the 48-item assessment Mathematical Meanings for Teaching Secondary Mathematics (MMTsm). A major goal of Project Aspire was to provide information to professional development leaders. We tried to write descriptions of rubric levels that would capture certain ways of thinking, without requiring that the scorer be familiar with the nuances of those ways of thinking. The Project Aspire team and the BEAR team at UC Berkeley ran several rounds of inter-rater reliability (IRR) and used the results to refine the items and rubrics.

The second author translated each item into Korean. A Korean high school mathematics teacher who was a mathematics Ph.D. student translated the items back into English. The second author and the third author reviewed the back translations and the second author made adjustments to the Korean versions (Behling & Law, 2000; Harkness, Van de Vijver, Mohler, & fur Umfragen, 2003). We collected U.S. teacher data in 2014 and 2015 from multiple professional development settings and scored by the Project Aspire team, with some overlapping scores with which to run IRR. The Korean data was collected in the summer of 2015 and scored by English-speaking Korean teachers that tested sufficiently high on the rubrics after training. The second author then scored a subset of responses to run IRR.

Figure 1 shows the Willie Chases Robin task; this paper analyzes responses for Parts B and C. Willie Chases Robin "is a frame of reference context where an individual uses one clock to time two events that begin at different times…Thus, when an individual uses both times in the same expression and in the same unit, she must offset one time from the other to account for the differences in *elapsed* time. In addition, the item's references to times are in two different units—speed (distance relative to time) measured in miles per hour, and the difference in their elapsed time measured in minutes".

Robin Banks ran out of a bank and jumped into his car, speeding away at a constant speed of 50 mi/hr. He passed a café in which officer Willie Katchim was eating a donut. Willie got an alert that Robin had robbed the bank, jumped into his patrol car, and chased Robin at a constant speed of 65 mi/hr. Willie started 10 minutes after Robin passed the café.

Part A. Let u represent the number of hours since Robin passed the café. Write an expression that represents the number of hours since Willie left the café.

Part B. Here are two functions. They each represent distances between Willie and Robin.

$$f(x) = 65x - 50\left(x - \tfrac{1}{6}\right), x \geq 0.$$

$$g(x) = 65\left(x - \tfrac{1}{6}\right) - 50x, x \geq 1/6.$$

i) What does x represent in the definition of f?

ii) What does x represent in the definition of g?

Part C. Functions f and g both give a distance between Willie and Robin after x hours. But $f(1)=6.67$ and $g(1)=4.17$. Why are $f(1)$ and $g(1)$ not the same number?

Figure 1. Willie Chases Robin MMTsm Item. ©2014 Arizona Board of Regents. Used with permission.

We then scored the teacher results with the rubrics in Figure 2 and Figure 3.

B4 Response:	The teacher said <u>both</u> of the following things: - x in $f(x)$ represents number of hours (or elapsed time) since Willie left café - x in $g(x)$ represents number of hours (or elapsed time) since Robin left café
B3 Response:	Matches B4 response except that x in $g(x)$ is since Robin left <u>bank</u>
B2a Response:	Matches B4 response except no reference points (café, bank) mentioned
B2b Response:	Matches B4 except teacher switched meanings for x in $f(x)$ and in $g(x)$
B1 Response:	Teacher gave same meanings for x in $f(x)$ as in $g(x)$
B0 Response:	The response doesn't fit a higher level, cannot be interpreted, has no clear answer, or is off-topic, but isn't blank or just the statement "I don't know".

Figure 2. Willie Chases Robin Part B MMTsm rubric. ©2014 Arizona Board of Regents. Used with permission.

Part B of the Willie and Robin item (see Figure 1) aims to see whether teachers would interpret the meaning of parts of function definitions by analyzing them quantitatively and with explicit reference to their domains. The highest level for this item, B4, is for responses where the teacher distinguished between both independent variables by the reference point of their magnitudes. The only way for two non-equivalent functions' definitions to represent the same quantity (distance between the men) is for the independent variable in each to have different meanings, which is why responses that said both x's have the same meaning were placed at the level B1. Levels B3, B2a and B2b were for responses that articulated the difference to some degree but did not specify the exact quantitative meaning of the x's. Figure 2 summarizes our rubric for Part B.

C2 Response:	Teacher said $f(1)$ and $g(1)$ represent distance between men at two different moments in time, or made same statement for x=1 in $f(x)$ and in $g(x)$.
C1 Response:	Teacher said x=1 has different meanings in both functions but a) did not elaborate on the meaning of x, b) described both x's as representing distances, or c) described $f(1)$ and $g(1)$ as representing time passed; or, described $f(1)$ and $g(1)$ as representing distances but not specifically distances between men.
C0 Response:	The response doesn't fit a higher level, cannot be interpreted, has no clear answer, or is off-topic, but isn't blank or just the statement "I don't know".

Figure 3. Willie Chases Robin Part C MMTsm rubric. ©2014 Arizona Board of Regents. Used with permission.

Part C of the Willie Chases Robin item (see Figure 1) is designed to see whether teachers could articulate why two very functions could represent the same quantity yet have different values for the same independent value. The answer, as in Part B, is that the meaning for x in each function is different. For example, if Robin passed the café at 4:00pm, then the distance between the two men at 5:00pm is given by either $f(1)$=6.67 and $g(1.16)$=6.67. Variables (and quantities) have no useful meaning without specified reference points from which we are measuring. Figure 3 summarizes our rubric for Part C.

Earlier we said that our theory therefore functions as an explanatory framework for how people think about quantities. By writing item-specific rubrics that described precisely what types of responses belong to each level, we sought to create rubrics that could categorize teachers' meanings for frame of reference without requiring the scorer to fully understand the theory of what constitutes a conceptualized frame of reference. Our item and rubrics can then be used to either assess the needs of a particular group of teachers for teaching, research, or professional development purposes, or to function as pre- and post- items to evaluate the efficacy of an instructional intervention.

Results & Discussion

In this section we discuss what individual responses can tell us about the teacher's meanings for quantities within a frame of reference, by studying several representative examples through the lens of our theoretical framework. Korean responses were translated into English and handwritten by the second author, and the country of origin of each sample response is not identified (gender pronouns were selected randomly).

Willie Chases Robin Part B Results

Part B elicited a wide range of responses, and so we built a rubric that looked at all three of the commitments necessary to fully conceptualize a frame of reference: unit, reference point, and directionality of comparison. Figure 4 displays three teacher responses to Part B.

Part B. Here are two functions. They each represent distances between Willie and Robin.	$f(x)=65x-50\left(x+\frac{1}{6}\right), x\geq 0.$ $g(x)=65\left(x-\frac{1}{6}\right)-50x, x\geq 1/6.$	
i) What does *x* represent in the definition of *f* ? x represent the number of hours since Willie left the cafe	i) What does *x* represent in the definition of *f* ? the number of hours Robin passed the café	i) What does *x* represent in the definition of *f* ? Related to Robin given he is 1/6 hr ahead of Willie
ii) What does *x* represent in the definition of *g*? x represents the number of hours since Robin passed the cafe	ii) What does *x* represent in the definition of *g*? the number of hours Katchim passed the café	ii) What does *x* represent in the definition of *g*? Related to Willie given that he is 1/6 hr behind Robin

Figure 4. Teacher responses that were scored at a) Level B4, b) Level B2b and c) Level B0.

The response in Figure 4a was scored at the highest level of B4 because of three aspects we deemed important, all of which allow us to build a hypothetical model of how the teacher was reasoning while answering this item. The teacher clearly specified “number of hours” and so was identifying each *x* as representing a quantity; other responses merely referred to “time” which could apply equally to the passage of time or the time of day. The teacher also specified reference points and used the appropriate reference points (leaving the café for both men) to make sense of the function definitions. Without reference points for a quantity’s measurement, there is no clear unambiguous relationship between a given measurement and the quantitative situation it represents. Finally, the teacher correctly identified that *f* gave the distance between the two men in terms of Willie’s time since leaving the café, where *g* is in terms of Robin’s time since leaving the café. In order to correctly identify each function’s independent value, the teacher had to reason about how one would adjust each man’s time in terms of the others to calculate his distance from the café, in terms of his speed times the number of hours he drove. Our model for how this teacher reasoned was that she conceptualized the quantity with an internal commitment to unit, reference point, and directionality of comparison.

The response in Figure 4b was scored at Level B2b because it is identical to a B4 response *except* that the teacher reversed the meanings of the *x* in the definition of *f* and the *x* in the definition of *g*. The definitions he gave do not allow for *f* and *g* to represent the distance between Willie and Robin. The teacher’s response is consistent with using one directionality of comparison to define each measurement of time, but the opposite directionality of comparison to define each man’s time in the other man’s frame of reference. Our model for how this teacher reasoned about Part B was that he conceptualized the quantity with an internal commitment to unit and reference point; we hypothesize that instead of committing to a directionality of comparison the teacher took the +1/6 in (x+1/6) to indicate a later time and therefore a description of Willie’s behavior. In this case, the increase is seen as an indicator of “largeness”, instead of an indicator that *x*’s value needs to be augmented to represent the appropriate meaning within this frame.

The response in Figure 4c was scored at Level B0 because it did not fit any higher levels, and we can see why when we look at this response in terms of the commitments the teachers did and did not make. This teacher identified the difference in the x's by a general indication that each one has something to do with one person in the context and referred to the difference of 1/6 hours in starting time between the two men. We can see that the teacher is hinting at something relating to the difference in reference points for each man's measurement of time, but she does not know how to interpret that difference by defining two quantities with different reference points. Our model for how this teacher reasoned is that she did not define either x in terms of any quantities (precise or vague) at all, so she made no commitments to unit, reference point, or directionality of comparison in this response.

Willie Chases Robin Part C Results

Part C was particularly difficult for teachers from both countries, as shown in Table 1 in the next section. In deciding how to differentiate responses in a meaningful way, we decided that the most valuable information from Part C responses was in how the teachers did or did not commit to a reference point. Therefore, our rubric for Part C is built around assessing commitment to a reference point. Figure 5 displays three teacher responses to Part C.

Part C. Functions f and g both give a distance between Willie and Robin after x hours. But $f(1)$=6.67 and $g(1)$=4.17. Why are $f(1)$ and $g(1)$ not the same number?		
F(1)=distance between 2 men when officer has left cafe & traveled 1 hr g(1)= distance between 2 men since Robin has travel 1 hr beyond cafe-Cop is catching up	f(1) represent the distance Robin has gone in 1 hour. g(1) represent the distance the Officer has gone in 1 hour.	Because Willie started $\frac{1}{6}$ hrs at the café, the distance between two people should be represented by g(x). f(1) is fixed at café where Willie is.

Figure 5. Teacher responses that were scored at a) Level C2, b) Level C1 and c) Level C0.

The response in Figure 5a was scored at the highest level of C2 because this teacher described f(1) and g(1) as both representing the distance between the two men, but at different points in time because of the different meanings of x in each function. The prompt in Part C sets up a seeming contradiction and asks the teacher to reason why the contradiction does not, in fact, exist. To do so, this teacher had to think about the quantitative meaning of the independent value x in each function, and realize that different reference points for the inputs necessarily implied different meanings for the dependent values as well. Our model for how this teacher reasoned is that he conceptualized all four quantities x [in $f(x)$], x [in $g(x)$], $f(x)$ and $g(x)$ with commitments to reference points.

The response in Figure 5b was scored at Level C1 because this teacher described $f(1)$ and $g(1)$ as representing distances at different points in time, but not specifically distances between men. To reach this conclusion, she had to keep her commitment to the definitions of each x, but not make the same conclusions about the dependent values as the teacher in Figure 5a. Our model for how this teacher reasoned is that she conceptualized x [in $f(x)$] and x [in $g(x)$] with commitments to reference points, but not $f(x)$ or $g(x)$.

The response in Figure 5c was scored at Level C0 because it did not fit any higher levels, and we can see why when we look at how this teacher was not able to resolve the seeming contradiction posed to him. We do not have enough information to speculate about how he conceptualized the quantities represented by x in each function, but we can conclude that he did not conceptualize the quantities $f(x)$ or $g(x)$ with commitments to reference points.

Scores For All Teacher Responses

Table 1 shows a breakdown of responses to Part B and Part C, with 186 US and 365 Korean teachers responding. I/X responses consisted solely of "I don't know" or were blank. We do not make any quantitative analyses or conclusions in this paper, but we find the distributions of interest to those who want to see how large samples of teachers answer this question. Though the purpose of our paper is not a comparison between countries, we must note that when we only had US data, many objections were raised to Project Aspire papers and presentations by saying that our items were too difficult and therefore inappropriate to be given to secondary teachers. Our Korean data shows that this is not necessarily true.

Table 1. Responses to Willie Chases Robin. ©2014 Arizona Board of Regents. Used with permission.

	Part B Results	
	US	Korea
B4	23 (*12.4%*)	144 (*39.5%*)
B3	5 (*2.7%*)	8 (*2.2%*)
B2a	5 (*2.7%*)	73 (*20.0%*)
B2b	35 (*18.8%*)	25 (*6.8%*)
B1	29 (*15.6%*)	33 (*8.8%*)
B0	81 (*43.0%*)	74 (*20.0%*)
I/X	9 (*4.8%*)	10 (*2.7%*)

	Part C Results	
	US	Korea
C2	12 (*6.4%*)	56 (*15.3%*)
C1	24 (*12.9%*)	139 (*38.1%*)
C0	140 (*75.3%*)	160 (*43.8%*)
I/X	10 (*5.4%*)	10 (*2.7%*)

Conclusion

While other professional development projects continue to administer the MMTsm, the data discussed here shows that our theoretical framework is useful in analyzing teachers' reasoning with frames of reference. Our rubrics, correlated to different levels of productive meanings for quantities within a frame of reference, allow us to analyze teacher responses to our tasks and characterize to what extent each teacher reasoned about quantities within frames of reference on our tasks. The MMTsm was designed specifically to investigate mathematical meanings *for teaching*; we are interested in modeling the kinds of meanings that teachers might convey to students in their classroom. While a teacher with productive meanings for quantities within frames of reference is not guaranteed to help her students develop productive meanings, it is certainly true that a teacher with unproductive meanings will have difficulty in doing so.

One limitation of our data is that the teachers were pulled from voluntary participants in professional development programs (for the US) and voluntary participants taking exams mandatory for teachers finishing their fifth year of teaching (in Korea). Neither population is representative of their country as a whole. However, our non-random results do suggest that many teachers are probably not prepared to help their students reason through such tasks. It is important that mathematics and mathematics education professors are aware of teachers' weak meanings for frame of reference and address them during undergraduate instruction and professional development settings. We cannot begin to address a problem until we have identified it, and teacher reasoning with frames of reference is an important yet heretofore unidentified area in need of further study and intervention.

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