The Role of Multiplicative Objects in a Formula

Irma E. Stevens
University of Georgia

The goal of this article is to propose a way to think about the role of a multiplicative object in reasoning about formulas quantitatively and covariationally. Building off the works of others on the importance of constructing multiplicative objects when reasoning about graphical representations, I adapt their definitions to be able to include a meaningful way to discuss what it means to construct a multiplicative object with a formula. I then use the analysis of six sessions of a semester-long teaching experiment with a preservice secondary mathematics teacher to illustrate what it means not to construct and what it means to construct a multiplicative object with a formula.

Keywords: Cognition, Precalculus, Preservice Teacher Education

One of the upcoming avenues of research in the quantitative reasoning literature is studying the role the construction of a multiplicative object has in a meaning for a graph "as a continuum of states of covarying quantities" (Saldanha & Thompson, 1998) (e.g., Frank, 2016, 2017, in press; Stevens & Moore, 2017; Thompson, 2011; Thompson & Carlson, 2017). In this paper, I build on the research done with graphical representations by discussing the role constructing a multiplicative object has in a meaning for a symbolic representation (namely, a formula) that represents the varying measures of attributes identified in a situation. I propose a way to conceive of a multiplicative object with a formula. I then demonstrate the role of conceiving of a multiplicative object when constructing a formula to represent quantities in a situation. To do so, I will use the results of a four-month long individual teaching experiment designed to support a preservice secondary mathematics teacher's covariational reasoning and construction of formulas through dynamic geometric environments.

Background

What is a Multiplicative Object?

The notion of a multiplicative object first stemmed from "Piaget's notion of 'and' as a multiplicative operator—an operation that Piaget described as underlying operative classification and seriation in children's thinking" (Thompson & Carlson, 2017, p. 433) (e.g., Inhelder & Piaget, 1964; Piaget, 1970). Frank (2017) described Inhelder and Piaget's notion of a multiplicative relationship as schemas that invoke an image of simultaneity. The general idea is for an individual to construct a new attribute that simultaneously incorporates two other identified attributes. For example, Frank (2017) noted that a person can conceive of objects that are red, objects that are circular, and simultaneously, objects that are red circles. The final object is a uniting of the two other attributes, and thus, involves a multiplicative operator.

Saldanha and Thompson (1998) extended the idea of multiplicative objects by discussing it in terms of quantities (i.e., measureable attributes). For Saldanha and Thompson (1998), a multiplicative object involves constructing pairs of values. They described it as entailing a coupling of two quantities so that "one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value" (p.

299). In 1990, Thompson defined a quantity's value as "the numerical result of a quantification process applied to it," which at that time to him meant that either "direct or indirect measurement" was taking place. He has since updated his definition of quantification (see Thompson, 2011), but in that update, he did not offer a new definition for a value. Thus, I offer an updated definition of values that is rooted in the understanding of quantities' magnitudes as Wildi magnitudes (see Thompson, 2011; Thompson, Carlson, Byerley, & Hatfield, 2014; Wildi, 1991). He argued that a quantity's magnitude (or amountness) is invariant of the unit used to measure it. I argue that the amount is the same regardless of the unit, but the value of a quantity necessarily depends on the unit chosen to measure it. Thus, a quantity's value refers to an obtained or anticipated measure of a magnitude using a defined unit magnitude for the quantity. The resulting measure is expressed numerically.

For the sake of clarity, when I refer to magnitudes, I refer to students' images of quantities or unmarked bars representing the students' conception of that quantity's amountness (e.g., the red and blue bars in Figure 1) without explicit attention towards units. When I refer to values, I refer to measurements (using either assumed or anticipated units) expressed numerically or symbolized within formulas. Thus, I update Saldanha and Thompson's (1998) definition of a multiplicative object by replacing "value" with "magnitude" in order to distinguish between reasoning about quantities vs. measurements. That is, a multiplicative object entails a uniting of objects so that one tracks a quantity's magnitude with the immediate, explicit, and persistent realization that, at every moment, the other quantity (quantities) also has (have) a magnitude(s).

Researchers have primarily discussed multiplicative objects in the context of graphing activities (Frank, 2016; (Frank, 2016; Stevens & Moore, 2017; Stevens, Paoletti, Moore, Liang, & Hardison, 2017). Frank (2016) discussed how to conceptualize a point in the Cartesian coordinate system as a multiplicative object. Figure 1 shows two quantities' magnitudes represented on a pair of axes. The plotted point on the graph is the result of the uniting of the two quantities. Thus, each point on the graph represents the magnitudes of two quantities simultaneously. The result can be expressed as values in a coordinate pair using (x, y). Students do not always interpret a point in a Cartesian plane as representing a multiplicative object (Frank, 2017; Stevens & Moore, 2017), and in this paper, I will demonstrate that the difficulty of representing multiplicative objects extends into reasoning with formulas as well.

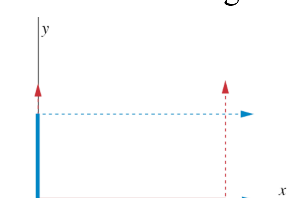

Figure 1. Frank's (2016) image of a projection of two quantities' magnitudes represented on axes and then projected to construct a single coordinate pair.

Why is a Multiplicative Object Important in Covariational Reasoning?

Based on the definition of a multiplicative object, there is an understanding that as two quantities' magnitudes covary in Figure 1, the resulting location of the point will change with it. How students reason about the covarying of the two quantities is split into six levels of covariational reasoning (Thompson & Carlson, 2017, p. 441). Covariational reasoning, in general, occurs when students conceive of situations as composed of quantities that vary in tandem (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002), and researchers have deemed it important to understanding ideas about rate of change (Ellis, 2007; Johnson, 2015; Oehrtman, Carlson, &

Thompson, 2008; Thompson, 2011). A student cannot be classified in the top three levels of covariational reasoning if she has not constructed a multiplicative object. In Frank's (in press) study of interviews from three pre-calculus students, she noted how the two students who engaged in emergent shape thinking (i.e., constructing a graph as an emerging representation of a covariational relationship) attended to the quantities' values represented on the axes (i.e., the blue and red bars) as a way to help them conceptualize two attributes uniting. They represented this uniting by constructing a coordinate pair (x, y). She and others have concluded about the importance of supporting students in organizing images of varying quantities to construct meaningful representations. In the following section, I discuss how the idea of a multiplicative object is relevant to the construction of symbolic representations; namely, formulas.

To illustrate an example of the process of constructing a multiplicative object and then reasoning covariationally, I use the city task in Saldanha and Thompson (1998) in which students "engaged in a sequence of tasks centered around the activity of tracking and describing the behavior of the distances between a car and each of two cities as the car moves along the road" (p. 300) (Figure 2). In looking at the situation, the two quantities in the situation are highlighted using dotted line segments; namely, they are the respective distances the two cities are from the car. The quantities are represented as perpendicular magnitudes to the left of the image, isolated from the remainder of the situation. By identifying unit lengths for the quantities, it is possible to construct a Cartesian coordinate system by partitioning along the magnitudes and beyond. (In doing so, the student also has a unit magnitude identified with which to produce values.) The point P in this Cartesian coordinate system now represents the correspondence of the magnitudes of the distances between both cities. Based on Thompson and Carlson's (2017) levels of covariational reasoning, this correspondence is the first evidence that a multiplicative object has been constructed. To reason covariationally with this newly constructed multiplicative object, the student must anticipate changes in the magnitudes situated on the axes resulting in changing the correspondence point (i.e., the multiplicative object) as the car travels. If the point P is traced, a graph relating the two quantities emerges. For more details on the construction of the graph, see Moore and Thompson (2015).

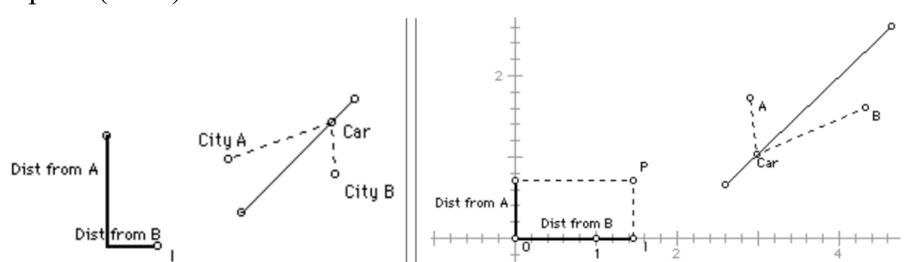

Figure 2. Saldanha and Thompson's (1998) image of the City Travels Problem.

How Does Constructing a Multiplicative Object Support Quantitative and Covariational Reasoning with Formulas?

The previous example motivates a need for students to unite cognitively two quantities' measures. For graphical representations, the purpose is clear; in order to construct a quantitative image of a graph, the student must construct a point P as a multiplicative object. What is unclear in the literature is how the role of a multiplicative object plays a role in either reasoning with formulas.

Consider a known formula: $A = ½bh$, a commonly presented formula for the area of a triangle. Students first use this formula in the 6^{th} grade "to find the area of right triangles,

triangles... by composing into rectangles or decomposing into triangles and other shapes" (National Governors Association Center for Best Practices, 2010). To do so, students identify a measure for a base, *b*, and its corresponding height, *h*, to calculate the measure for the area of the triangle, *A*. In this context, there is no intellectual need for a multiplicative object because there is no variation in the quantities. However, consider the case in which the measurement of the triangle's height varies. Then, simultaneously, the measurement of the triangle's area also varies. To use formulas to represent this covariation of quantities, a student must be able to unite the values of both the height and area of the triangle within the formula so that the united image of the quantities persists through the variation.

There are a few difficulties to consider when constructing a multiplicative object of a formula rather than a graphical representation. First, there is no single object within the representation that simultaneously represents the two quantities' measures as there is in a coordinate system. Rather, the uniting is an anticipation the student has that for any given instance of a triangle, there is a single pair of values (assuming the student has established units) to represent that instantiation. Secondly, and relatedly, there are no magnitude bars present in a formula; in a Cartesian graph, a student can identify magnitudes representing the values of quantities on the pairs of axes, and these magnitude bars change as the values for the quantities change. For a formula, however, glyphs (i.e., symbolic inscriptions) represent the values of the magnitudes of the quantities and these symbols do not alter as quantities in the situation vary. It is left to the student to have a meaning for those symbols that enables them to anticipate changing quantities' values in either their image of the situation, of corresponding magnitude bars, or a sequence of numbers that the individual can imagine running through (Oehrtman et al., 2008). Lastly, in the same way that a situation has quantities a student has to push to the background of their mind so that they can instead focus only the two quantities under consideration, in a formula, the student must isolate the symbol (or group of symbols) that represent the two quantities under consideration. Figure 3 illustrates this idea by using colors to bring attention to the two quantities (height and area) that the student attempts to reason about covariationally (Figure 3). One can imagine the colors shifting to different quantities represented both in the image and the formula as the student conceptualizes varying different pairs of quantities.

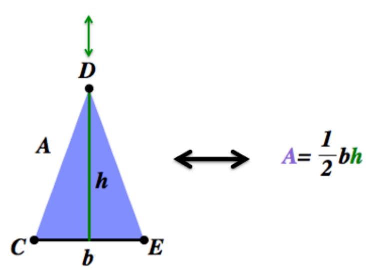

Figure 3. Constructing a multiplicative object between the height and area of a triangle using the formula ***A***= ½ b***h***.

Methods

I explored how students construct and use multiplicative objects with formulas as part of a semester-long teaching experiment with three undergraduate students in a preservice secondary education mathematics program at a large public university in the southeastern U.S. The reason I chose preservice teachers is because of their vast mathematical experiences and their commitment to understanding secondary mathematics ideas through their undergraduate study. During the study, these students were enrolled in a course based on the *Pathways Curriculum* (Carlson, O'Bryan, Oehrtman, Moore, & Tallman, 2015) in which they learned about quantitative and covariational reasoning. Each student participated in 12-15 teaching sessions, totaling 18-19.5 hours of interview time per student. I video-recorded and screen-captured students' work on a tablet and made scans of student work. At least one observer was present at all but one interview. During and after each interview, we took notes of students' activities and planned future teaching sessions. Throughout the sessions, the importance of constructing and using multiplicative objects emerged, and thus, the analysis for this portion of the study focused on students' development of that idea by coding videos of the data. In this particular study, I focus on Lily's meanings for her formulas through the theoretical lens of her construction and use of a multiplicative object with two known formulas. I also attended to her levels of covariational reasoning based on Thompson and Carlson's (2017) framework. I limited the analysis to Lily's first six interviews because it was in these interview that she was working on problems with familiar formulas and first constructing multiplicative objects. I conducted a conceptual analysis (Steffe & Thompson, 2000) so that I could develop second order models of her thinking.

Task Design

In the first sessions of the teaching experiment, I updated a task based on the results of a previous study with preservice secondary teachers (Stevens, 2018). The task consisted of three parts, one given at each of the first three teaching sessions, each with the same starting prompt: "How would you describe the relationship between the height and area of isosceles triangles?" I particularly limited them to isosceles triangles in an attempt to limit the images students could have of what it would mean to vary the height of the triangle. In the first part, I gave the prompt without any other associated image. In the second part, I asked the question with a given static triangle. In the third part, I asked them to consider what would happen if the height of the triangle changed, providing them with a sketch created with dynamic geometry software in which they could drag one of the vertices of the triangle to change its height (as in the triangle in Figure 3 but without the green segment visible).

Lily started working on the Painter Problem in her fourth interview. This problem is similar to the growing rectangle problems other researchers have used (Ellis, 2011; Kobiela, Lehrer, & VandeWater, 2010, May; Matthews & Ellis, in press; Panorkou, 2017). In this problem, I ask the student to "relate the length that Kent [who is painting a wall in his home] has pulled the paint roller and the area that he has covered in paint" (Figure 4).

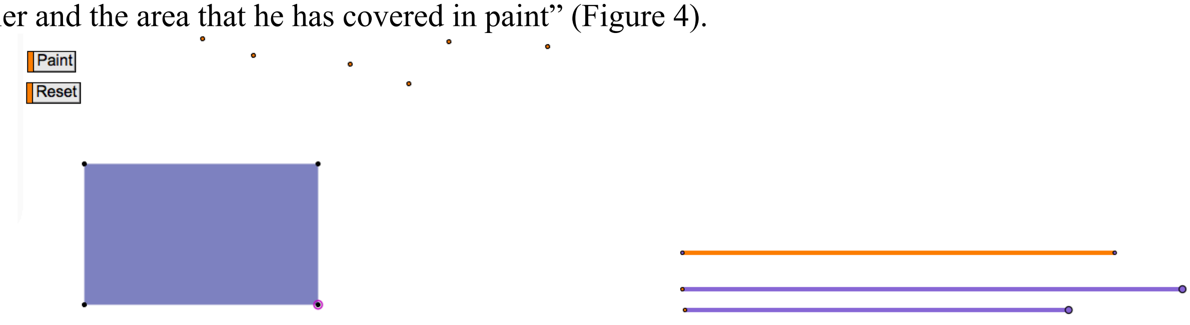

Figure 4. The Painter Problem.

Results

In the following section, I report on the results of the teaching sessions. The results are split into two parts characterize students' thinking as it relates to constructing and using multiplicative objects as it relates to formulas. First, I describe instances in which Lily did not construct a multiplicative object with her formula and then I describe Lily's first construction of a multiplicative object with her formula in the Painter Problem.

No Multiplicative Object Constructed in a Formula

One of the main aforementioned components of constructing a multiplicative object is to isolate two quantities. In the Triangle Problem, students are asked to consider the relationship between two quantities, the height and area of a triangle. Lily, when given this prompt, quickly identified the formula for the area of a triangle as the normative $A= \frac{1}{2} bh$. However, she struggled to continue to relate the height and area because of the presence of the b in the formula. She wanted to express the relationship symbolically with only A and h symbols present. The following transcripts show evidence of this reasoning. The parentheses beside the name indicate which interview number the statement occurred.

> *Lily* (1): This is the area formula. So we know that our area is 1/2 -- area of any triangle is 1/2 base, height. But this is asking for the relationship between the height and the area, so the base is kind of like a -- I mean, I guess I'm trying to say that it's like not explicitly just between the height and the area, and the base is like in that [formula].
>
> *Lily* (2): I [*sees image of static triangle*] – [*pause*] That's my triangle. [*pause*] Area. [*pause*] But I want to relate just the area to just the height, so I need to get rid of that [*b in her formula*]. Not get rid of it, but write it in terms of area and the height, because I'm specifically trying to relate [*sighs*] area and the height. So I'm going to do -- I didn't want to write A. [*pause*] Obviously area equals area but it blows my mind.

As illustrated in the two transcript excerpts above, Lily wanted to use a formula to represent the relationship between the two quantities. However, she was dissatisfied with the presence of a third quantity, b, in her formula. In the second excerpt, she attempted to "get rid" of it by solving for b in her area formula and then re-substituting it into the formula. This resulted in her writing $A=A$. She was not satisfied with the outcome because then only one quantity, A, remained in her formula, rather than A and h.

Lily's reasoning here is an example of the importance of understanding that the relationship between two quantities might be influenced be a third quantity (or more), and that the presence of a symbol representing that quantity in a formula does not exclude that formula from representing the relationship between the two quantities under consideration. Because her meaning for formulas entailed an understanding that a formula relates all the quantities present in a formula, Lily did not isolate two quantities to construct a multiplicative object with her formula in the way illustrated in Figure 3.

Coordinating Values Between Quantities as Evidence for Construction of a Multiplicative Object

When given the Painter Problem in her fourth interview with the bars, Lily tried to solve for h (the length of the paintbrush) in her formula $A=bh$ (A= area painted, b is length rolled) as she did in the Triangle Problem. However, she suddenly switched to consider how her formula could be used to describe the directional covariational relationship she identified in the situation (i.e., as the length swept out increases, the area painted increases). She pointed to the b and A in her formula [*underlined in Figure 5*], wrote down the calculations on the bottom right of Figure 5 and stated the following:

> *Lily* (4): Yeah, the height stayed constant and we just changed the base [*motioning along the orange highlighted base in Figure 5*], and as it got bigger, the area got bigger [*pointing to the results of her area calculations, 5 and 10*]. Just because there's more space too, that he painted [*motioning along rectangle*]. Like if you stop here [*draws in dotted line*], [*focusing on the rectangular image*] the base would be smaller and there's not as much of an area. But if it gets bigger, there's more of an area. This got bigger [*motioning along the*

orange highlighted base]. This Dimension. And the amount [*motioning along rectangle*] got bigger as well.

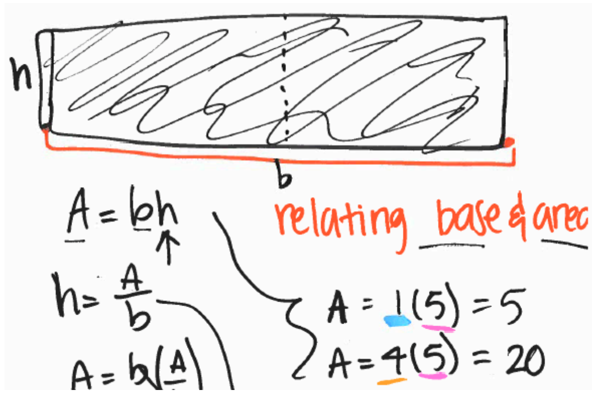

Figure 5. Lily's initial activity on the Painter Problem.

In order to isolate the two quantities in a formula, Lily connected how the quantities were changing or staying constant in the situation with her formula. That is, she noted that the height stayed constant in the situation, so *h* now also represented a constant (i.e., 5) to her. This idea enabled her to focus on what was changing, the base, and so she was able to consider different values for *b*. She then noted the results of her calculations as varying values for *A*, which she connected back to her situation by discussing "more space." Thus, her construction of a multiplicative object when she was able to coordinate different pairs of values for *h* and *b* with a connected image of how those different values corresponded to the quantities in the situation. It is important to note that Lily's activity here does not demonstrate that she envisioned changes in quantity's values, and so she can only be said to have a coordination of values rather than images of covariation. In fact, evidence for her using her constructed multiplicative object to reason in a way in which she could connect reasoning about amounts of change in her situation with values in her formula were not present until the end of her sixth interview. However, her coordination of values here is example of the first level to include the construction of a multiplicative object.

Conclusions and Discussion

Lily's activity over the course of six interviews demonstrated how her meaning for a formula developed as she was able to construct a multiplicative object within her formula that she could connect to her understanding of the dynamic situation. For her, writing the two calculations in Figure 5 was crucial to her conceptualizing the formula as able to represent pairs of values between two quantities using one formula that contained symbols for quantities that she was not trying to relate. I argue that providing Lily with a dynamic situation helped support her in accommodating her meanings for formulas in a way that enabled her to isolate quantities in her formula and construct a multiplicative object. Overall, I argue that in the same way that multiplicative objects are important for covariational reasoning within graphical representations, it is also important for symbolic representations, particularly formulas.

Acknowledgments

This paper is based upon work supported by the National Science Foundation under Grant No. DRL-1350342. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF. I would also like to thank Kevin Moore for his helpful feedback on this study.

References

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378.

Carlson, M. P., O'Bryan, A. E., Oehrtman, M. C., Moore, K. C., & Tallman, M. A. (2015). *Pathways algebra and precalculus* (5 ed.). Kansas City, KS: Rational Reasoning.

Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction, 25*(4), 439–478.

Ellis, A. B. (2011). Generalizing-promoting actions: How classroom collaborations can support students' mathematical generalizations. *Journal for Research in Mathematics Education, 42*(4), 308–345.

Frank, K. (2016). Plotting points: Implications of "over and up" on students' covariational reasoning. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 573–580). Tucson, AZ: The University of Arizona.

Frank, K. (2017). *Examining the development of students' covariational reasoning in the context of graphing.* (PhD), Arizona State University.

Frank, K. (in press). The relationship between students' covariational reasoning when constructing and interpreting graphs *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education*. San Diego, CA.

Inhelder, B., & Piaget, J. (1964). *The early growth of logic in the child* (E. A. Lunzer & D. Papert, Trans.). New York, NY: Norton & Company, Inc.

Johnson, H. L. (2015). Together yet separate: Students' associating amounts of change in quantities involved in rate of change. *Educational Studies in Mathematics*, 1–22. doi:10.1007/s10649-014-9590-y

Kobiela, M. A., Lehrer, R., & VandeWater, E. (2010, May). Students' developing conceptions of area via partitioning and sweeping *Paper presented at the 2010 annual meeting of the American Educational Research Association*. Denver, CO.

Matthews, P. M., & Ellis, A. B. (in press). Natural alternatives to natural number: The case of ratio. *Journal of Numerical Cognition*.

Moore, K. C., & Thompson, P. W. (2015). *Shape thinking and students' graphing activity.* Paper presented at the Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education, Pittsburgh, PA: RUME.

National Governors Association Center for Best Practices, C. o. C. S. S. O. (2010). Common Core State Standards for mathematics: Grade 6 Geometry.

Oehrtman, M., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. L. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27–42). Washington, D.C.: Mathematical Association of America.

Panorkou, N. (2017). Dynamic measurement: The crossroad of area and multiplication. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 339–346). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Piaget, J. (1970). *Genetic Epistemology*. New York: W. W. Norton & Company, Inc.
Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen, K. R. Dawkings, M. Blanton, W. N. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the 20th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 298–303). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.
Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–307). Hillside, NJ: Erlbaum.
Stevens, I. E. (2018). Insights into students' images of a geometric object and its formula from a covariational reasoning perspective *Proceedings of the Twenty-First Annual Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) Conference* (pp. 997–105). San Diego, CA.
Stevens, I. E., & Moore, K. C. (2017). The intersection between quantification and an all-encompassing meaning for a graph. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 709–716). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.
Stevens, I. E., Paoletti, T., Moore, K. C., Liang, B., & Hardison, H. (2017). Principles for designing tasks that promote covariational reasoning. In A. Weinberg, C. L. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 928–936). San Diego, CA: RUME.
Thompson, P. W. (1990). *A theoretical model of quantity-based reasoning in arithmetic and algebra*. Retrieved from Center for Research in Mathematics & Science Education:
Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e* (pp. 33–57). Laramie, WY: University of Wyoming.
Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). Reston, VA: National Council of Teachers of Mathematics.
Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In L. P. Steffe, K. C. Moore, L. L. Hatfield, & S. Belbase (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing* (pp. 1-24). Laramie, WY: University of Wyoming.
Wildi, T. (1991). *Units and conversion charts: A handbook for engineers and scientists*. New York: IEEE Press.