Analyzing Topology Students' Schema Qualities

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A schema is a mental structure of concepts that are connected together and allows for the efficient functioning of director systems. Skemp (1979) discusses various qualities that this study used to look at students' schemas. This case study focuses on a pair of Topology students and their work on a problem involving the product topology on $X \times Y$. There were many positive qualities that the students demonstrated, but there were also difficulties with particular connections between concepts.

Keywords: Topology, schema, director system

Theoretical Background

Topology is an important course for advancement in mathematics. Many graduate students need to take topology to continue in their mathematics degrees. Regrettably, research on pedagogy of topology is still in its infancy. In a study by Berger and Stewart (2018), the analysis of the data revealed that the majority of undergraduate students were in the beginning stages of schema development, even though they were completing a final examination at the end of their semester. Cheshire (2017) looked at axiomatic structures of Topology and how students' schemas undergo accommodation to understand these structures.

Understanding mathematical concepts at an advanced level is an enormous undertaking for many students. The word 'understanding' reminds us of the well referred work by Skemp (1976) on *relational understanding* and *instrumental understanding*. Some years later, Skemp (1979) developed a model of intelligence in which its focus was the construct of the idea of *schema*. In Skemp's (1979) notion:

> A schema is a structure of connected concepts. The idea of a cognitive map is a useful introduction, a simple particular example of a schema at one level of abstraction only, having concepts with little or no interiority, and representing actuality as it has been experienced. A schema in its general form contains many levels of abstraction, concepts with interiority, and represents possible states (conceivable states) as well as actual states. (p. 190)

A schema is what allows for the efficient functioning of a *director system*, which is a central focus to Skemps' model. A person or object has a present state that they are currently in, and a goal state that they would like to be in. "That which is changed from one state to another and kept there" (p. 41) is what Skemp calls the *operand*. The *operator* is "that which actually does the work of changing the state of the operand." (p. 41) Finally, a director system is "that which directs the way in which the energy of the operator system is applied to the operand so as to take it to the required state and keep it there." (p. 41-42)

Skemp (1979) communicated his theoretical ideas through many everyday examples. He referred to the temperature of an oven in many instances. Say an oven is at room temperature and needs to heat to 400 degrees Fahrenheit. The present state is the current temperature and the goal state is to reach 400 degrees Fahrenheit. The operand is the interior of the oven and the operator is the temperature of the oven. The thermostat in the oven is the director system. If there is a new goal state of 350 degrees Fahrenheit, the same director system (the thermostat) will be used.

A schema is what gives a director system this flexibility when states change. "The greatest adaptability of behavior is made possible by the possession of an appropriate schema, from which a great variety of paths can be derived, connecting any particular present location to any required goal location." (Skemp, 1979, p. 169) Skemp's work with director systems was also used by Olive and Steff (2002, p. 106) to build "a theoretical model of children's constructive activity in the context of learning about fractions."

Skemp (1979) believed that "a schema is a highly abstract concept" (p. 167). Some of his qualities of schema and the definitions of certain words that are used are shown in Table 1.

Table 1. Certain qualities of a schema.

Qualities of a schema	Definitions
(ii) "Relevance of content to the task in hand (rather obviously, but not always met)." (p. 190)	
(iii) "The extent of its domain." (p. 190)	Domain: "The set of states within which (and only within which) a director system can function, i.e., can take the operand to its goal state and keep it there, provided that the operators are capable." (p. 312)
(iv) "The accuracy with which it represents actuality." (p. 190)	
(v) "The completeness with which it represents actuality within this domain." (p. 190)	
(vi) "The quality of organization which makes it possible to use the concepts of lower or higher order as required, and to interchange concepts and schemas. (The vari-focal part of the model, linked with the idea of interiority.)" (p. 190)	Vari-focal: "A way of describing the different ways in which the same concept or schema can be viewed, from a simple entity to a complex and detailed structure." (p. 316)
(vii) "By a high-order schema, we mean one containing high-order concepts…This determines its generality…" (p. 190)	
(viii) "The strength of the connections." (p. 190)	
(ix) "The quality of the connections, whether associative or conceptual." (p. 190)	
(x) "The content of ready-to-hand plans…" (p. 191)	Plan: "A path from a present state to a goal state, together with a way of applying the energies available to the operators in such a way as to take the operand along this path. A plan is thus one essential part of the director system." (p. 314)

The notion of schema has also been explored by others in the literature. For example, a definition of schema is embedded in APOS Theory (Dubinsky & McDonald, 2001). They claimed that "a schema for a certain mathematical concept is an individual's collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept." (p. 277)

In this paper we will examine students' development of their schemas and their qualities based on Skemp's (1979) model. The research question to guide this study was: What qualities of schema do Topology students demonstrate?

Methods

In this case study, we examined students' schemas for a basis for a topology. This is part of a larger study for the first author's dissertation. The participants were first year graduate students who were enrolled in a graduate topology course at a Southwestern University in the U.S. The four participants were divided into two pairs. Each pair did two task-based interviews together, the first about a month into the semester, and the second during finals week of the same semester. Pairs were used to try to get the participants to demonstrate their ideas and discuss how they think about the tasks to their partner, making their thoughts more observable. No data was collected regarding what took place in the classroom before, after, or between these interviews.

The participants were given a task sheet and a definition sheet. Each interview began with a period of time where the participants could look through and work on the tasks individually. After that, they worked on the tasks as a pair, explaining their thoughts to each other and coming to a consensus for each problem. In the final part of each interview, the pairs were asked follow-up questions about what they thought was needed to complete each task. They were also asked about their background with Topology. In the interviews that occurred during finals week, they were additionally given their work from earlier in the semester and asked to discuss their progress between then and finals. Each interview was video recorded and then transcribed. If the participants utilized the white boards, their written work was also transcribed. In the transcriptions, a scribble indicates that the pair erased something on the board.

For this study, we focused on the third task only (see Figure 1). We chose to analyze the data from the third task for a few reasons. First, this problem is one that frequently shows up on homework assignments and exams when the product topology is covered in class. As such, we have been able to collect data in the past involving this same problem, regardless of what Topology course the data was collected from. This problem also requires a higher-order schema for a basis and, because of this, we hypothesize that the data will be more informative about certain qualities of schema needed. After transcribing the data from this task, we established some themes from Skemp's model based on the qualities and created Table 2 as a framework. After creating an ideal proof with ideal qualities, we examined our data against it.

3. (a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. Define the product topology $\mathcal{T}$ on $X \times Y$.

   (b) Show that the projection map $p_X : X \times Y \to X$ defined by $p_X(x, y) = x$ is an open map.

Figure 1. Task 3. Show that the projection map is an open map.

Results and Discussion

The step by step proof of part (b) has been illustrated in Table 2. The lower-order concepts needed for this proof, as well as the schema qualities ideal for completing each portion of the proof, are shown. We acknowledge that there are more qualities of schema that can be applied in each step, however, the listed qualities are what we focused on based on our experience with this problem and our data set. After being transcribed, the data was divided up by what portion of the proof it aligned with. In each portion, we provide some explanation of the data in terms of Skemp's qualities of schema. Ideally, learners' schemas become more structured as they go throughout the course, but this study is not solely focused on comparing the early and final interviews. Instead, we are looking for changes in students' schemas and what qualities are involved in those changes.

Table 2. Qualities of each portion of task 3.

Portion of proof	Proof for part (b)	Explanation of Proof Step	Lower-order Concepts Needed	Qualities of Schema
b1	Consider $p_X(W)$ where $W \subset X \times Y$ is an arbitrary open set of $X \times Y$.	Start with an arbitrary open set of $X \times Y$ and see where p_X sends it.	-Open map	-Plan -Domain -Relevance
b2	Now $p_X(W) = p_X(\cup_\alpha (U_\alpha \times V_\alpha))$ where $U_\alpha \times V_\alpha \in \mathcal{T}$ are basis elements.	By the definition of a basis, W can be written as a union of basis elements.	-Topology generated by a basis -Equality of sets	-Strength -Quality -Domain
b3	Note $p_X(\cup_\alpha (U_\alpha \times V_\alpha)) = \cup_\alpha p_X(U_\alpha \times V_\alpha)$.	The projection of a union is a union of projections.	-Projection map -Equality of sets	-Generality -Domain -Accuracy -Strength -Quality
b4	Now $\cup_\alpha p_X(U_\alpha \times V_\alpha) = \cup_\alpha U_\alpha$ where $U_\alpha \in \mathcal{T}_X$.	The projection map sends basis elements to open sets of X.	-Projection map -Definition of the product topology on $X \times Y$	-Accuracy -Completeness
b5	Since $\cup_\alpha U_\alpha \in \mathcal{T}_X$, $p_X(W) \in \mathcal{T}_X$ and p_X is an open map.	The union of open sets of X is also open in X.	-Topology -Open map	-Strength -Vari-focal

Initial Interview

For the purposes of this paper, we focus on only one of the pairs of participants: Brandon and Kyle (pseudonyms). Brandon had not completed a Topology course before the initial interview and Kyle had previously taken an introductory Topology course, so their experience with the subject matter was limited. Additionally, this pair was more interactive with each other and took their time discussing each task.

In their initial interview, Brandon and Kyle defined the product topology without using a basis and then had a short proof based on their incorrect definition. This was the typical mistake that was found in previous work with undergraduate students (Berger & Stewart, 2018). Specifically, in defining the product topology in part (a), the pair incorrectly stated that the product topology consisted of all sets of the form $U \times V$, not the unions of such sets. Therefore, they set themselves up for a fairly trivial proof for part (b). They both took some time wrapping their minds around the problem, Kyle in more of a verbal manner. For b1, they took their arbitrary open set to be exactly what is expected based on their response to part (a), which is $U \times V$ where $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$ (see Figure 2). Their director system functioned appropriately, but their previous knowledge led them to start part (b) at the wrong present state.

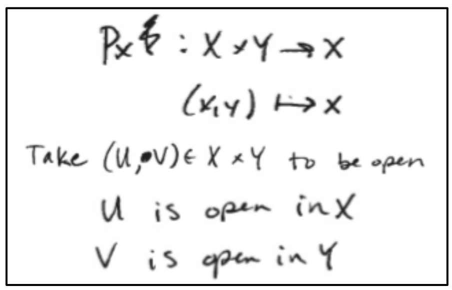

Figure 2. Brandon and Kyle's attempt early in the semester.

From there, they followed the definition of the projection map and immediately got what they needed in order to show that the map is open (see Figure 3). Since this step was a fairly straightforward computation for the pair, their ideas were accurate and complete, but only relative to their incorrect definition in part (a). With regards to their schemas for a basis, we cannot make any claims since their work here provided no evidence regarding a basis.

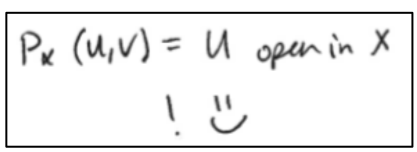

Figure 3. The end of Brandon and Kyle's proof.

Final Interview

Now we will discuss what Brandon and Kyle did during finals week. For part (a), they correctly defined the product topology by generating it with a basis. For b1, Brandon quickly wrote an arbitrary open set, W, on the board, but neither Brandon nor Kyle discussed it (see Figure 4). Brandon had a plan and executed it without discussing it with Kyle. Mathematically, this was relevant and fit within the domain of the problem.

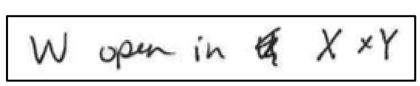

Figure 4. The start of Brandon and Kyle's proof late in the semester.

For b2, Brandon began writing W as a union of basis elements, but Kyle seemed unsatisfied and wanted to make the notation clearer. Brandon asked how they wanted to proceed in showing the union. Kyle took over writing on the board, erased the union that Brandon had written, and came up with the beta notation shown in Figure 5. After this, he was still unsatisfied with his notation and the usage of too many b's, but Brandon said it was fine, so they moved on. Note that Kyle immediately knew that they were wanting to write a union of basis elements, but the pair spent their time here struggling to denote it. Towards the beginning of their discussion on b2, Kyle stated "W is the union of elements of...some arbitrary union of elements of that B thing [referring to the basis they wrote in part (a)]." Kyle's statement demonstrates a strong conceptual connection between open sets and a basis. Again, this was appropriate within the domain of the problem.

$\Rightarrow W =$

$\bigcup_{B \in \beta} B$ for some $\beta \subset \mathcal{B}$

Figure 5. Where Brandon and Kyle had notational difficulties.

In moving on to b3, they started off quickly saying that the projection gives you a union of open sets, but then Kyle starts thinking about if they could get "weirder things". This launched the pair into a discussion about what they showed in class regarding the projection map. Brandon finally says something that takes them back to what they originally (and correctly) said, which prompted Kyle to read the task again and agree that they had been on the right track. Although their discussion may have been helpful for them in checking their ideas, it ended up not affecting the proof that they wrote down as the discussion was entirely verbal and ended with the same conclusions that they began with. They discuss what exactly the projection does, and this prompts Kyle to suddenly write up the proof seen in Figure 6. They verbally acknowledge that they get a union of open sets of X from the basis elements and use this as the end of their proof.

Take W open in $X \times Y$

$W = \bigcup_{B \in \beta} B \qquad \beta \subset \mathcal{B}$

$= \bigcup_{U \times V \in \beta} U \times V$

$U \times V \in \beta \Rightarrow U$ open, V open

$\pi_X(W) = \pi_X\left(\bigcup U \times V\right)$

$= \bigcup U$ open in X

Figure 6. Final part of Brandon and Kyle's proof.

The part where the pair got off topic demonstrates a loss of relevance to the task and not as strong of connections between the projection map and unions. At the beginning and end of b3, however, they did demonstrate accuracy in their statements about what the projection map does. They combined b4 and b5 of the proof with their b3 statements and did not make any conclusion statements for their proof. Their final statements from b3 are accurate, but do not demonstrate complete ideas. We cannot say anything about what qualities they demonstrated for b4 or b5 since they did not say or write any concluding remarks.

Towards the end of the final interview, the pair was asked to reflect on their work from the initial interview. After reviewing their previous work, Brandon and Kyle quickly confirmed that they had not considered a basis in the first interview. When asked about what could have aided in correcting that mistake, Brandon responded that what corrected it for him was getting feedback on his homework "…with that specific thing being torn apart on it." Kyle admitted that he didn't "get the basis topology stuff at all when [they] were going over it in class" but later realized the importance when reading through the textbook. Brandon commented on their earlier work on the problem with,

> In terms of when we first did this, I guess, um, it's easy to just jump straight into just choosing U or something because of the way we defined the basis of just being an open set cross an open set where each of those are coming from the individual…so…it just seems natural to just go to one thing instead of considering the most general thing, which is a union of those things.

Brandon's reflection suggests that the generality and relevance of a schema are not necessarily intuitive.

Concluding Remarks

In this study we saw that accuracy and completeness are not typically a difficulty, but rather the generality of a schema, strength of connections, and the relevance of a schema can be difficult to navigate. Both Brandon and Kyle demonstrated strong, conceptual connections between lower-order concepts and a basis in the final interview, but not in the early interview. In the final interview, the pair demonstrated a weaker connection between the projection map and unions.

Employing Skemp's model in order to develop a more general theoretical framework to examine learner's schemas are among our next steps. Future work will also involve analyzing the other pair of participants, as well as analyzing the remainder of the tasks. Additionally, we will be interviewing graduate students whose research area is in Topology, as well as postdoctoral fellows to examine their schema qualities.

Based on this work, some teaching recommendations could include emphasizing scenarios where arbitrary objects and generality are necessary in higher-order topics, focusing on giving good feedback for students regarding what qualities are missing in their work, and explicitly discussing conceptual connections between concepts frequently.

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