The Interaction Between a Teacher's Mathematical Conceptions and Instructional Practices

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This study raises questions about a common assumption that an advanced degree in mathematics is sufficient for teaching courses in undergraduate mathematics meaningfully. The study reports results from 24 mathematics PhD students' solutions to a precalculus level problem requiring quantitative reasoning. We also describe the PhD students' conceptions of what knowledge is needed to produce a meaningful solution to this task. These graduate students' problem solving approaches and images of the reasoning abilities needed to solve the problem were classified as having either a static calculational orientation or a dynamic conceptual orientation. We share how these two orientations are exhibited in the context of teaching precalculus students. We further illustrate ways in which a teacher's actions to support her students in conceptualizing and relating quantities led to her engaging her students in more dynamic conceptually oriented discussions.

Keywords: quantitative reasoning, mathematical knowledge for teaching, teaching practice

Mathematics departments across the nation assign incoming mathematics PhD students teaching assignments in precalculus and beginning calculus. In US mathematics departments these assignments are often made based on the students' prior coursework in mathematics and their ability to communicate clearly in English. Some mathematics departments provide teaching workshops for their incoming PhD students in mathematics. However, it is common for these workshops to focus primarily on the mechanics of teaching, with little or no focus on what the mathematics education research literature has revealed about the processes of learning or teaching ideas in the courses they are assigned to teach (Ellis, 2015). It has also been reported that graduate students in mathematics sometimes have weak understandings of fundamental ideas they are expected to teach (Musgrave & Carlson, 2016). These PhD students are typically offered little support in considering what is involved in understanding or learning the key ideas of courses they are assigned to teach; nor are they supported in determining how or whether to engage students during class, what to include in a lecture, how to assess student learning. Given the background and experiences of these new mathematics instructors it is likely that their instructional decisions and actions will be based on such things as their current conceptions of the mathematics they teach and their experiences in learning these ideas as students (Stigler & Hiebert, 1999).

In this study we investigated the mathematical approaches that incoming PhD students in mathematics used when completing a standard applied problem (see *Figure 1*) in a course in precalculus. We also probed their view of the knowledge they used to complete the problem. Subsequently two of these 24 students participated in weekly professional development aimed at supporting precalculus teachers in engaging their students in developing stronger meanings of the ideas of precalculus and improved ability to access these ideas when confronting novel problems. The analyses of these two teachers' classroom videos reveal stark differences in the teachers' images of how students' understandings develop. They also highlight instructional practices that led to students' constructing stronger meanings. The results of this study may also provide new directions for preparing mathematics PhD students for teaching.

Literature Overview and Theoretical Framing

Over 30 years ago Shulman (1986) encouraged research to pay more attention to the knowledge base that teachers need to carry out the practice of teaching. He called for increased attention on what he called pedagogical content knowledge, "the ways of representing and formulating the subject that make it comprehensible to others" and greater understanding of what makes student learning of specific topics difficult (Shulman, 1986, p. 9). Even and Tirosh (1995) further called for teachers to develop understandings of student ways of thinking and suggested that this knowledge should inform the activities they use to engage students. One such reconceptualization was the introduction of the construct of mathematical knowledge for teaching (MKT) in which pedagogical and mathematical knowledge were combined into one category (e.g., Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008; Hill et al., 2008). Silverman and Thompson's (2008) study of teaching placed greater focus on the mathematical understandings, how they are connected, and how a teacher might spontaneously leverage these understandings when teaching. They also call for teachers to ponder how these understandings might develop in the minds of students (Silverman & Thompson, 2008, p. 500). Silverman and Thompson (2008) later proposed that developing MKT involves transforming a teacher's personal understandings of a mathematical concept to an understanding of how this understanding might be useful for students' learning of related ideas. They call for teachers to be supported in developing their images of the kinds of activities and conversations that might support another person in developing an understanding of an idea. They advocate that teachers try to envision learning the concept as a student and keep this in mind when developing activities to use with his students. By imagining scenarios from the viewpoint of a student, a teacher is better prepared to guide and direct conversations with his students.

Other lines of inquiry into teaching have considered what teachers pay attention to and how they respond to student utterances in the context of teaching. Some of these constructs include calculational orientation, teacher noticing, and decentering. Thompson, Philipp, Thompson, and Boyd (1994) characterized and contrasted a teacher exhibiting a calculational orientation when interacting with students with one who exhibited a conceptual orientation when conversing with students about their approach to working an applied problem in a 7th grade classroom. They illustrate questions posed by a teacher that is oriented more toward helping students understand why an approach works (e.g., can you explain why that calculation makes sense?), and contrasted these to questions (e.g., what do you do next?) asked by a teacher that was focused on students' completing the calculations to get the correct answer.

Jacobs, Lamb, and Philipp (2010) studied teaching by examining children's strategies, interpreting their understanding, and deciding how to respond on the basis of children's understanding. They call these integrated abilities "professional noticing" and claim that they enable teachers to make appropriate instructional decisions based on student thinking. Thompson (2000), Steffe (1990), and Moore and Carlson (2012) leveraged Piaget's (1955) idea of decentering to characterize the quality of teacher-student discourse. In Piaget's work on children's cognitive development, he introduced the idea of decentering to describe a child's transition from his or her egocentric thought to the capability of adopting the perspective of another. As a teacher shifts to consider a student's perspective and expressed meanings she is said to be attempting to decenter (Thompson, 2000).

Conceptualizing Quantities in a Problem Context

In recent years many researchers have found Thompson's (1990, 1994, 2011) idea of quantitative reasoning to be fundamental to working applied problems in precalculus and calculus (e.g., Engelke, 2007; Moore & Carlson, 2012). Thompson claims that when students process the words in an applied problem they should be conceptualizing the measureable attributes of objects that are described in the problem context. According to Thompson a quantity does not exist in the world; rather a quantity is constructed in the mind of an individual when she imagines measuring some quality of an object, such as a person's height or the person's distance from home as she drives to work (Thompson, 2011). A quantity's value is the numerical measurement that a quantity assumes. When the value of a quantity is static it is called a constant or fixed quantity. If the value of a quantity changes throughout a situation it is referred to as a varying quantity. A quantitative operation occurs in the mind of an individual when conceptualizing a new quantity in relation to one or more already-conceived quantities (Thompson, 2011). When one conceives of three quantities related by means of a quantitative operation, he has conceptualized a quantitative relationship. One is said to be engaging in quantitative reasoning when he is actively engaged in constructing a network of quantities and quantitative relationships (Thompson, 1988, 1990, 2011).

Context of the Study

The *Pathways to Transforming Undergraduate Mathematics Education* project supports future mathematicians (PhD students in mathematics) to develop as reflective teachers who leverage research on student learning and formative data to adapt their instructional practices. The PhD students in the program attend a 3 day workshop prior to teaching with research based instructional materials, and then attend a weekly seminar during each semester that they teach a course using these materials. The materials include cognitively scaffolded in-class investigations that engage students in quantitative and covariational reasoning as cross-cutting ways of thinking that lead to students' understanding and using the course's ideas. Detailed instructor notes and solutions illustrate both productive and unproductive student thinking relative to specific ideas.

Method

The data presented in this study is from a larger study that followed 2 PhD students from a pool of 24 incoming mathematics PhD students over the first two years of their teaching precalculus at a large public university. Upon their entering the program they and 22 other students completed 5 mathematics problems to assess their conceptions of fundamental ideas of precalculus. Two of the 24 PhD students who were assigned to teach pre-calculus were subsequently video-taped when teaching during their first 3 semesters of teaching precalculus in the context of using a research based curriculum and attending weekly professional development meeting based in research on student learning, and designed to foster growth in the instructor's mathematical conceptions of precalculus ideas and how they are learned. The written responses of 24 incoming PhD students were analyzed relative to their: (a) conceptualization of the quantities in the problem context; (b) their usage and meanings for variables, terms and expressions; (c) their image of the transformation of the box; (d) the degree to which the box's transformation influenced their image of the constrained covariation of the two varying quantities to be related. We analyzed classroom video data of two teachers during their third semester in the Pathways TUME program. The lessons analyzed for this report had a focus on conceptualizing quantities in the context of the familiar box problem (see *Figure 2*). This video data was analyzed relative to the same four criteria used to analyze the written responses. In addition we analyzed the teachers' actions (utterances, drawings, questions, etc.) to glean insights about their approaches for supporting their students in engaging in quantitative reasoning, and their conceptions of how students might acquire the ideas that were central to the lesson.

Toy Chest Problem

An 8-foot by 4-foot piece of plywood is being used to build an open-top toy chest. The chest is formed by making equal-sized square cutouts from two corners of the plywood (see *Figure 1*). We remove these squares and make three folds (illustrated as dashed lines on the figure) to form three sides of a box. We then attach the three-sided box to the wall, so that we get an open top toy chest. Define a function f that determines the volume of the toy chest (in cubic feet) in terms of the length (number of feet) of one side of the square cutout, x.

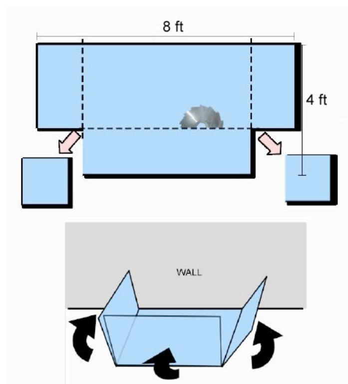

Figure 1. The toy chest problem

Results

The toy chest problem was one of the five problems that 24 PhD students in mathematics completed during an initial teaching workshop that took place during the summer prior their beginning their graduate studies. This problem asked student to define a function to determine the volume of a toy chest given the side-length of equal-sized squares that are cut from two corners of a plywood board. The problem was illustrated in a drawing with the dimensions of the plywood labeled and dashed lines indicating where the cuts could be made (see *Figure 1*).

Analysis of the responses of the 24 PhD students responding to this task revealed that only 13 of 24 of these students produced a correct response of $f(x) = (8-2x)(4-x)(x)$. The majority (7 of the 11) who produced an incorrect answer responded by writing $f(x) = (8)(4)(x)$. This response suggests that these mathematics graduate students were not imagining the sides of the toy chest varying with x, the length of the sides of the squares cut from the two corners. Instead they appeared to imagine a fixed length and width for the box, and failed to recognize how the box's length and width would vary as the value of the side-length of the squares varied. Other incorrect responses included, $f(x) = (8 - x)(4 - x)$ and $f(x) = (8 - 2x)(4 - 2x)(x)$, also suggesting that the symbols they produced were not based in an accurate image of the quantitative relationships described in the problem context.

In a follow up prompt these same PhD students were asked to describe how they would explain what it means to solve the equation $f(x) = 9$, and how they would support students in understanding what it means to evaluate $f(3)$ and solve the equation $f(x) = 9$. The PhD students' responses included: (a) one is finding x and the other is finding $f(x)$ so I would show them how to calculate these values when the other value is known; (b) solving $f(x) = 9$ using algebra might be too hard for them, but they should have no problem finding the point that has a y-value of 9; (c) when evaluating $f(3)$ you are putting 3 in for x and finding a value for the box's volume. When solving $f(x) = 9$ you are putting in a value for the box's volume and finding a value for x. The first 2 responses (typical of over half to the 24 subjects) focus on what students should do to answer the questions, with no mention of the quantities represented by the symbols or what it means to evaluate a function or solve an equation. In contrast, the third response includes references to the quantities and describes what the process of "evaluating" and "solving for" produces in terms of the quantities in the situation. A stronger response (not provided by any of the incoming PhD students) might also convey that evaluating a function for a particular value of the input quantity involves using the function rule or process to determine the corresponding value of the output quantity. Solving $f(x) = 9$ would then be described as producing a value of the input quantity x as an instance of reversing the process of f, or determining a value for the square's side-length, x, when the box's volume is known. This data provides evidence of weaknesses in these graduate students' conceptions of a function, also suggesting that the majority of these PhD students viewed a function formula as a tool for determining values.

The Teaching of Jack and Gloria

The video excerpts of Jack and Gloria are presented to contrast two teachers' conceptions of a mathematics lesson that required their students to use quantitative reasoning to relate two varying quantities. Recall that this data was collected during the third semester in which Jack and Gloria were teaching precalculus in the context of the Pathways TUME project.

Jack's conceptions operationalized during teaching. Jack began his lesson with a picture of an 8.5" by 11" sheet of paper with squares 2 inches on each side removed from each of the four corners (see *Figure 2*). He had labeled one side of one of the four squares with the label 2".

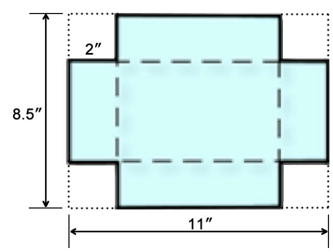

Figure 2. Jack's illustration of the box

He began his discussion of this problem by saying, "What I have drawn out on the board is an 8 and ½ by 11 inch piece of paper, out of which we have cut 2 inch squares." (Jack assumed the students understood that the squares removed from all four corners all had side lengths of 2 inches). He followed by saying, "We are going to fold the paper along these dashed lines." (The students are expected to imagine a paper being folded). He then said, "What we're interested in is the volume of this box when a 2 inch square is removed from each corner." He goes on to tell students that the volume of the box is length times width times height, but then follows by saying that "We're not going to worry why this formula works." but invites them to think about this on their own time. The lesson continues with him doing almost all of the talking while focusing on the calculations needed to determine the length, width, and height of the box that he has drawn.

Jack labeled the fixed quantities of 11" and 8.5" on his drawing (see *Figure 2*) and placed a 2 above one of the squares. It is noteworthy that he failed to make clear whether he was speaking about the square's area or the square's side length when referencing the 2. He followed by asking students what in the picture represented the length of the box. A student who appeared confused raised her hand and said in an inquiring way, "So the length would not be 11." The teacher followed by saying, “That is correct, the length will not be 11, but we may want to use 11 to determine the box’s length later on; just hold that thought.” This response suggests that the teacher was not interested in how the student was conceptualizing the quantities in the situation, rather he seemed more focused on what he wanted to say next. Jack then moved on to ask the same student what the width would be. She responded similarly in an inquiring tone, “So the width wouldn’t be 8.5 (pause), would it be 2 something?” The teacher did not respond to her question, but again points to the box’s width in his illustration on the board. He followed by calculating each dimension of the box while writing $(11-(2)(2))(8.5-2(2))(2)$ and concluded the discussion by saying, “We could use the same method to find a box with a different square. Right?”

This exchange suggests that Jack was not interested in how the student was conceptualizing his drawing. When the student asked why the box’s length was not 11” Jack took no action to support her in conceptualizing the box’s length; nor did he pose questions to support this student in visualizing how the box’s length varied with changes in the side of the square. His description of how to calculate the box’s length and width suggests that he believed that writing and saying these calculations conveyed an image of how the box’s width and the square’s side length are related. He did not appear to be interested in how he was being interpreted and did not show interest in his student’s thinking. His questions direct students’ attention to a static image of the paper and were focused more on what calculations to use to compute a volume.

Gloria’s conceptions operationalized during teaching. Gloria’s discussion of the box problem began with her providing pairs of students with an 8.5” by 11” sheet of paper, scissors, and tape. On the overhead projector were instructions to build a box by cutting four equal sized squares from the corners, and folding up the sides. As she circulated from table to table she challenged students to build a box that would hold only a small amount of popcorn and others to build a box that would hold the largest amount of popcorn possible. After the students had built their boxes she held up four boxes and asked her students to vote on which box would hold the most popcorn. Her choice to have students build the box suggests that Gloria recognized the need for students to take time to initially conceptualize quantities in the problem context and to consider how they are related. After the students had built their boxes, Gloria asked students to discuss what quantity in the situation determined each box’s shape. After a few minutes of discussion, students expressed a consensus that each box’s shape depended on the side length of the squares cut from the corners. Gloria followed by displaying a Geogebra animation she had developed prior to class (This applet allowed her to vary the side length of the squares while displaying how both the paper and box’s dimensions were transformed). Gloria began her discussion around the applet by saying, “Since we decided that the box’s shape and dimensions depend on the side length of the squares cut from the corners, let’s see what happens when we vary this quantity.” As she varied the side length steadily from 0 to its maximum value (4.25”) she asked students to describe how the box’s volume was changing. She interjected a prompt for students to explain what they were visualizing when thinking about the box’s volume. Students who responded conveyed they were visualizing such things as the amount of space inside the box and how much popcorn the box holds. She then asked her class what units they might use for measuring the box’s volume. After they discussed this with one another, she used a second applet that allowed her to vary the shape of the box, while displaying a varying number of cubes 1 inch on each side that would fit into the displayed box. As Gloria continued to vary the side length of the square cutout she prompted students to move their index finger upward from their desk to represent the box’s volume increasing and downward to represent the box’s volume decreasing. Students' first attempt to represent how the volume of the box was varying resulted in many students moving their finger upward only. Gloria called on one student to explain why she was moving her finger the way she did, and she replied that she was visualizing the height of the box getting taller and taller. Gloria asked the student to describe what attribute she was looking at when she was thinking about the box's volume. The student quickly recognized that she was paying attention to the wrong attribute of the box. Gloria again moved the side length continuously form 0 to 4.25, while all students moved their fingers upward to a point, and as the box became taller and narrower, they began to move their fingers downward until the paper folded onto itself. Gloria called on particular students to verbalize what they were imagining as they moved their fingers upward and then downward. She also asked particular students to describe the minimum and maximum values for the side length of the square that could be cut from the paper.

Gloria then had students work in their groups to complete a table to determine the value of the box's width, length, height and volume, given 4 values for the square's side length. While they were working she walked around the classroom as students completed the calculations and asked specific students to describe what their calculations represented in the context of the box's dimensions. Gloria posed a final question for students to determine an expression to represent the box's volume in terms of the side-length of the squares cut from the box's corners. While students were working she circulated around the class to ask students how they defined the independent variable and what quantity their expressions represented in the context of the box. In one case a student had written $2x - 11$ for the box's side length, instead of $11 - 2x$. Gloria asked this student to point to what x represented in the context of the box, what $2x$ represented in the context of the box and what 11 represented. Once the student had done this, the student noticed that his answer did not represent anything in the context of the box and he changed his answer to $11 - 2x$, explaining that 2 side-lengths are subtracted from 11 to get the box length. Gloria's persistent attention on her students' conceptions of the quantities in the problem context suggests that she believed that quantitative reasoning would enable her students to visualize what variables, expressions and formulas represent, and to see these symbols as representing how the box's volume varied with (or was related to) the length of the side of the square x that was cut from the box's corners.

Conclusions and Discussion

The data collected from the 24 incoming PhD students suggests that these highly successful mathematics students may have some of the same impoverished meanings and ways of approaching contextual problems as what has been reported in the literature about undergraduate students in mathematics. This finding suggests that even PhD students in mathematics might benefit from professional development focused on what is involved in understanding and learning ideas that are the focus of their instruction. The teaching episodes of Jack and Gloria contrast two conceptions of what is involved in supporting students in engaging in quantitative reasoning as a means for constructing formulas that represent how quantities in a problem vary together. Jack displayed a strong tendency to focus on static relationships and computations, while Gloria focused more on understanding ideas and visualizing quantities as they varied. Jack's interactions further reveal that he had little interest in understanding the meanings his students were constructing, while Gloria was regularly concerned with how students were conceptualizing a situation or representing a quantity. Her strong orientation toward her students' thinking and her actions to support students in constructing productive meanings led to many instances in which her interactions with her students led to advancements in their thinking.

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